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Network Flows, Semirings and Fuzzy Relations

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1 Introduction

In this work we will try to describe networkflows with methods of relational algebra. For this purpose we have a long way before us: first we deal with relations in classical sense, where our special attention is on the cardinality of relations under composition and meet (Dedekind inequality). After that we take a look at the so-called fuzzyrelations, which are “weighted“ relations, where a number between zero and one is a measure for the weight of a pair. If on these constructions composition and operations on sets like meet and join are defined in a suitable way the laws for clascic relations remain unchanged. Another important role will be played by semirings, by which we can prove properties of fuzzyrelations in an elegant manner.

The basis of our considerations is [Kaw], plus excursions in the world of semirings and particulary tests in semirings. These ideas we will introduce, because some theorems of [Kaw], especially those about flows, can be stated with tests and testrelations in a more compact and intuitive way.

2 Boolean Relations

2.1 Definitons

A relation α from a set X in a set Y we denote $\alpha : X \rightharpoondown Y$. Such a relation can be regarded as a subset of the cartesian product $X \times Y$. For reasons, which will be later explained, we call such a relation also a *boolean relation*. The *cardinality* $|\alpha|$ of a (boolean) relation $\alpha : X \rightharpoondown Y$ is the cardinality of the subset of $x \times Y$, which is defined by α.

However, relations are not isolated objects, we want to connect two or more relations. Therefore we define various operations on relatons.

Let $\alpha, \alpha' : X \rightharpoondown Y$ and $\beta : Y \rightharpoondown Z$ be relations. The *composition* $\alpha\beta : X \rightharpoondown Z$ is defined by

$$(x, z) \in \alpha\beta \Leftrightarrow \exists y \in Y : (x, y) \in \alpha \wedge (y, z) \in \beta.$$

The *join* $\alpha \sqcup \alpha'$ and the *meet* $\alpha \sqcap \alpha'$ of two relations α and α' is given by

$$(x, y) \in \alpha \sqcup \alpha' \Leftrightarrow (x, y) \in \alpha \vee (x, y) \in \alpha'$$

respectively

$$(x, y) \in \alpha \sqcap \alpha' \Leftrightarrow (x, y) \in \alpha \wedge (x, y) \in \alpha'.$$

The *converse* $\alpha^\sharp$ of a relation α is characterized by

$$(y, x) \in \alpha^\sharp \Leftrightarrow (x, y) \in \alpha.$$

A relation α^- we call the *complement* of α if

$$(x,y) \in \alpha^- \Leftrightarrow (x,y) \notin \alpha \wedge (x,y) \in X \times Y$$

holds.

By definition α is a relation from X into Y, and α^- is a relation from Y into X.

$id_X : X \rightharpoondown X$ denotes the *identity relation* on X with the property

$$(x,y) \in id_X \Leftrightarrow x = y,$$

$0_{XY} : X \rightharpoondown Y$ means the *zero relation*, corresponding with the empty set as subset of the cartesian product $X \times Y$ and $\nabla_{XY} : X \rightharpoondown Y$ denotes the *universal relation* with the property

$$(x,y) \in \nabla_{XY} \Leftrightarrow x \in X \wedge y \in Y.$$

If α', regarded as a subset of $X \times Y$, is contained in α, we write $\alpha' \sqsubseteq \alpha$. α is called *univalent*, if $\alpha^\sharp \alpha \sqsubseteq id_Y$, α is called *total*, if $id_X \sqsubseteq \alpha\alpha^\sharp$, and α is called *injective*, if $\alpha\alpha^\sharp \sqsubseteq id_X$. These algebraic characterizations are obviously equivalent to the common definitions of univalency, totality and injectivity.

A relation $\alpha : X \rightharpoondown X$ from X into itself is called a *endorelation*.

A total and univalent relation f from X into Y we call $function$. A function f is called *surjective*, if $f^\sharp f = id_Y$. This definition is as well equivalent to the common one.

2.2 Algebraic Properties of Operations on Relations

In the following section we will give some basic algebraic properties of operations on relations. Most of them are trivial; we won't mention them explicitly, if their application is not too sophisticated.

In the following α, β and γ are relations, so that the used operations are well-defined and meaningful.

- Commutativity, Associativity and Distributivity for $\sqcap$ and $\sqcup$:
  - $\alpha \sqcap \beta = \beta \sqcap \alpha$
  - $\alpha \sqcup \beta = \beta \sqcup \alpha$
  - $(\alpha \sqcap \beta) \sqcap \gamma = \alpha \sqcap (\beta \sqcap \gamma)$
  - $(\alpha \sqcup \beta) \sqcup \gamma = \alpha \sqcup (\beta \sqcup \gamma)$
  - $\alpha \sqcap (\beta \sqcup \gamma) = (\alpha \sqcap \beta) \sqcup (\alpha \sqcap \gamma)$
  - $\alpha \sqcup (\beta \sqcap \gamma) = (\alpha \sqcup \beta) \sqcap (\alpha \sqcup \gamma)$
- Absorption for $\sqcap$ and $\sqcup$:

- $\alpha \sqcup 0_{XY} = \alpha$
- $\alpha \sqcup \nabla_{XY} = \nabla_{XY}$
- $\alpha \sqcap 0_{XY} = 0_{XY}$
- $\alpha \sqcap \nabla_{XY} = \alpha$

- Properties of the Converse:
  - $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$
  - $(\alpha \sqcup \beta)^\sharp = \alpha^\sharp \sqcup \beta^\sharp$
  - $(\alpha \sqcap \beta)^\sharp = \alpha^\sharp \sqcap \beta^\sharp$
  - $(\alpha^\sharp)^\sharp = \alpha$
- Properties of the Complement:
  - $(\alpha \sqcup \beta)^- = \alpha^- \sqcap \beta^-$
  - $(\alpha \sqcap \beta)^- = \alpha^- \sqcup \beta^-$
  - $(\alpha^-)^- = \alpha$
  - $\alpha \sqcup \alpha^- = \nabla_{XY}$
  - $\alpha \sqcap \alpha^- = 0_{XY}$
- Combination of Complement and Converse:
  - $(\alpha^-)^\sharp = (\alpha^\sharp)^-$
- Associativity and Distributivity of Composition:
  - $(\alpha\beta)\gamma = \alpha(\beta\gamma)$
  - $(\alpha \sqcup \beta)\gamma = \alpha\gamma \sqcup \beta\gamma$
  - $\alpha(\beta \sqcup \gamma) = \alpha\beta \sqcup \alpha\gamma$
- Monotonicity:
  - $\alpha \sqsubseteq \beta \Rightarrow \alpha\gamma \sqsubseteq \beta\gamma$
  - $\alpha \sqsubseteq \beta \Rightarrow \gamma\alpha \sqsubseteq \gamma\beta$
  - $\alpha \sqsubseteq \beta \Rightarrow \alpha \sqcup \beta \sqsubseteq \alpha \sqcup \beta$
  - $\alpha \sqsubseteq \beta \Rightarrow \alpha \sqcap \beta \sqsubseteq \alpha \sqcap \beta$
  - $\alpha \sqsubseteq \beta \Rightarrow \alpha^\sharp \sqsubseteq \beta^\sharp$
  - $\alpha \sqsubseteq \beta \Rightarrow \beta^- \sqsubseteq \alpha^-$
  - $\alpha \sqsubseteq \alpha \sqcup \beta$
  - $\alpha \sqcap \beta \sqsubseteq \alpha$

These rules are almost all obviously and follow immediately from the definitions and elementary set-algebra. The only one we will closer look at is the distributivity of composition over join. The proof relies on the decisive spot on subtle connection between quantors and junctors: $(x,z) \in (\alpha \sqcup \beta)\gamma \Leftrightarrow$
$\exists y : ((x,y) \in (\alpha \sqcup \beta) \wedge (y,z) \in \gamma) \Leftrightarrow$
$\exists y : (((x,y) \in \alpha \vee (x,y) \in \beta) \wedge (y,z) \in \gamma) \Leftrightarrow$
$\exists y : (((x,y) \in \alpha \wedge (y,z) \in \gamma) \vee ((x,y) \in \beta \vee (y,z) \in \gamma)) \Leftrightarrow(!)$
$(\exists y : ((x,y) \in \alpha \vee (y,z) \in \gamma)) \wedge (\exists y : ((x,y) \in \beta \vee (y,z) \in \gamma)) \Leftrightarrow$
$(x,z) \in \alpha\gamma \vee (x,z) \in \beta\gamma \Leftrightarrow$
$(x,z) \in \alpha\gamma \sqcup \beta\gamma$ ■

One has to be careful, if $\sqcup$ is replaced by $\sqcap$: then in general only the following inequality holds:

$$(\alpha \sqcap \beta)\gamma \sqsubseteq \alpha\gamma \sqcap \beta\gamma$$

However, in [SchmStr, p.54 f.] is shown, that for univalent relations α the equality $\alpha(\beta \sqcap \gamma) = \alpha\beta \sqcap \alpha\gamma$ (analogously for injective relations γ the equality $(\alpha \sqcap \beta)\gamma = \alpha\gamma \sqcap \beta\gamma$) holds.

2.3 Cardinality of Boolean Relations

The cardinality of boolean relations has already been introduced. Trivial properties are $|\alpha| = |\alpha^\sharp|$ and $\alpha \sqsubseteq \alpha' \Rightarrow |\alpha| \leq |\alpha'|$. We will now investigate the behavior of the cardinality under composition and meet. The first fundamental property is the following:

Theorem 2.1 (Dedekind Inequality): *Let $\alpha : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ be relations. If α is univalent, the following inequalities hold:*

$$|\alpha \sqcap \gamma\beta^\sharp| \leq |\alpha\beta \sqcap \gamma| \quad \textit{and}$$
$$|\beta \sqcap \alpha^\sharp\gamma| \leq |\alpha\beta \sqcap \gamma|.$$

Proof: Choose an arbitrary $(x,z) \in \alpha\beta \sqcap \gamma$. Because of the definition of the composition of relations and the univalency of α there is an unique element $\alpha(x) \in Y$ so that $(x, \alpha(x)) \in \alpha$ and $(\alpha(x), z) \in \beta$ hold. We now look at the two mappings

$$\Phi : \alpha\beta \sqcap \gamma \rightarrow \alpha \sqcap \gamma\beta^\sharp \text{ und } \Psi : \alpha\beta \sqcap \gamma \rightarrow \alpha^\sharp\gamma \sqcap \beta,$$

defined by $\Phi(x,z) = (x, \alpha(x))$ and $\Psi(x,z) = (\alpha(x), z)$.
Next we observe, that Φ is surjective. To see this we take an arbitrary $(x,y) \in \alpha \sqcap \gamma\beta^\sharp$. Then (x,y) has to be contained both in α and in $\gamma\beta^\sharp$. Because $\gamma\beta^\sharp$ contains (x,y) a $z \in Z$ exists, so that $(x,z) \in \gamma$ and $(z,y) \in \beta^\sharp$, i.e., $(y,z) \in \beta$ hold. Therefore (x,z) is contained both in γ and in $\alpha\beta$; so we find for each $(x,y) \in \alpha \sqcap \gamma\beta^\sharp$ a $(x,z) \in \alpha\beta \sqcap \gamma$, which is mapped by Φ on (x,y). Similarly one

can obtain the surjectivity of Ψ. From the surjectivity of these mapping follow the inequalities above. ■

A relation $\alpha : X \rightharpoondown Y$ is called a *matching*, if $\alpha^\sharp\alpha \sqsubseteq id_Y$ and $\alpha\alpha^\sharp \sqsubseteq id_X$ hold. Equivalently we could demand, that both α and $\alpha^\sharp$ are univalent. Every injective function is obviously a matching.

With help of the Dedekind inequality we can show some properties of univalent relations and matchings:

Corollary 2.2: *Let $\alpha : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ be relations. Then hold:*

(a) If α uand β are univalent, then $|\alpha\beta \sqcap \gamma| = |\alpha \sqcap \gamma\beta^\sharp|$ holds.

(b) If α is a matching, then $|\alpha\beta \sqcap \gamma| = |\beta \sqcap \alpha^\sharp\gamma|$ holds.

(c) If α is univalent and β is a function then $|\alpha\beta| = |\alpha|$ holds.

(d) If α is a matching, then $|\alpha^\sharp\alpha\beta| = |\alpha\beta|$ and $|\beta\alpha\alpha^\sharp|$=$|\beta\alpha|$ hold.

Proof:

(a) To get familiar with the new way of thinking we will take a closer look at the proof of the first part.
Because α is univalent according to the Dedekind inequality

$$|\alpha \sqcap \gamma\beta^\sharp| \leq |\alpha\beta \sqcap \gamma|.$$

holds. Because for all relations the cardinality of a relation is equal to the cardinality of its converse, and because of $(\alpha\beta \sqcap \gamma)^\sharp = \gamma^\sharp \sqcap \beta^\sharp\alpha^\sharp$ we obtain

$$|\alpha\beta \sqcap \gamma| = |\gamma^\sharp \sqcap \beta^\sharp\alpha^\sharp|.$$

On the right side of this inequality we can because of the univalency of β apply the Dedekind inequality. This results in

$$|\gamma^\sharp \sqcap \beta^\sharp\alpha^\sharp| \leq |\beta\gamma^\sharp \sqcap \alpha^\sharp|.$$

By taking the converse $(\beta\gamma^\sharp \sqcap \alpha^\sharp)^\sharp = \alpha \sqcap \gamma\beta^\sharp$ we obtain the equality

$$|\beta\gamma^\sharp \sqcap \alpha^\sharp)| = |\alpha \sqcap \gamma\beta^\sharp|.$$

The summary of these equations and inequations can be written as

$$|\alpha \sqcap \gamma\beta^\sharp| \leq |\alpha\beta \sqcap \gamma| \leq |\alpha \sqcap \gamma\beta^\sharp|,$$

from which immediately follows the claim

$$|\alpha \sqcap \gamma\beta^\sharp| \leq |\alpha\beta \sqcap \gamma|$$

(b) Let α be a matching. Then holds:

$$
\begin{aligned}
|\beta \sqcap \alpha^\sharp\gamma| &\leq |\alpha\beta \sqcap \gamma| && \{ \text{ Theorem 2.1, } \alpha \text{ univalent } \} \\
&= |\gamma \sqcap \alpha\beta| && \{ \ \rho \sqcap \sigma = \sigma \sqcap \rho \ \} \\
&\leq |\alpha^\sharp\gamma \sqcap \beta| && \{ \text{ Theorem 2.1, } \alpha^\sharp \text{ univalent, } (\alpha^\sharp)^\sharp = \alpha \ \}
\end{aligned}
$$

(c) As a function β is total, therefore $\nabla_{XZ}\beta^\sharp = \nabla_{XY}$ holds. From $\alpha\beta \sqsubseteq \nabla_{XZ}$ follows $\alpha\beta \sqcap \nabla_{XZ} = \alpha\beta$. Hence we can conclude:

$$
\begin{aligned}
|\alpha\beta| &= |\alpha\beta \sqcap \nabla_{XZ}| && \{ \ \alpha\beta \sqcap \nabla_{XZ} = \alpha\beta \ \} \\
&= |\alpha \sqcap \nabla_{XZ}\beta^\sharp| && \{ \text{ part (a), } \alpha, \beta \text{ univalent } \} \\
&= |\alpha \sqcap \nabla_{XY}| && \{ \ \beta \text{ total } \} \\
&= |\alpha| && \{ \ \alpha \sqsubseteq \nabla_{XY} \ \}
\end{aligned}
$$

(d) Sei α ein Matching. Dann ist auch $\alpha^\sharp$ ein Matching, und es gilt:

$$
\begin{aligned}
|\alpha^\sharp\alpha\beta| &= |\alpha^\sharp\alpha\beta \sqcap \nabla_{YZ}| && \{ \ \alpha^\sharp\alpha\beta \sqsubseteq \nabla_{YZ} \ \} \\
&= |\alpha\nabla_{YZ} \sqcap \alpha\beta| && \{ \ \alpha^\sharp \text{ matching and part (b) } \} \\
&= |\alpha\beta| && \{ \ \beta \sqsubseteq \nabla_{YZ} \}
\end{aligned}
$$

The second equality follows from first simply by taking the converse. ■

An important role in the further course we play the singleton set $I = \{*\}$. It will server us as an “anchor“ for easier reasoning about sets or subsets and their cardinality, compare e.g. part (b) of the following corolarry. Obviously $id_I = \nabla_{II}$ and $\nabla_{XI}\nabla_{IX}$ hold for all sets X.

Corollary 2.3: *Let $\alpha : X \rightharpoonup Y$ and $\beta : Z \rightharpoonup X$ be relations. Then holds:*

(a) If f is a matching, then $|\nabla_{IX}f| = |f|$ holds.

(b) From $u \sqsubseteq id_X$ follows $|\nabla_{IX}u| = |u|$, particularly $|\nabla_{IX}| = |id_X| = |X|$.

(c) If f is an injective function, then $|\beta| = |\beta f|$ holds.

(d) If f is injective, then $|\nabla_{IX}| \leq |\nabla_{IY}|$ holds.

Proof:
(a) Let f be a matching. Then holds:

$$
\begin{aligned}
|\nabla_{IX}f| &= |f^\sharp\nabla_{XI}| && \{ \ |\alpha^\sharp| = |\alpha| \ \} \\
&= |f^\sharp| && \{ \text{ Cor. 2.2(c) with } f^\sharp \text{ matching and } \nabla_{XI} \text{ function } \} \\
&= |f| && \{ \ |\alpha^\sharp| = |\alpha| \ \}
\end{aligned}
$$

(b) Every subrelation $u \sqsubseteq id_X$ is obviously a matching, so (b) holds because of (a). The method to connect a set X with the relation ∇_{IX} will be very valuable for us.

(c) Let f be an injective function. Then holds

$$\begin{aligned}
|\beta| &= |\beta id_X| && \{ \text{ trivial } \} \\
&= |\beta f f^\sharp| && \{ id_X = f f^\sharp \} \\
&= |f f^\sharp \beta^\sharp| && \{ |\alpha^\sharp| = |\alpha| \} \\
&= |f^\sharp \beta^\sharp| && \{ \text{ Corollary 2.2(d) mit } f^\sharp \text{ matching } \} \\
&= |\beta f| && \{ |\alpha^\sharp| = |\alpha| \}
\end{aligned}$$

(d) Let f be injective. Then holds:

$$\begin{aligned}
|\nabla_{IX}| &= |\nabla_{IX} f| && \{ \text{ part (c) with } \beta = \nabla_{IX} \} \\
&\leq |\nabla_{IY}| && \{ \nabla_{IX} f \sqsubseteq \nabla_{IY} \} \qquad \blacksquare
\end{aligned}$$

2.4 Point Relations

So far we handled relations and their associated sets as a whole entity. To be able to talk about single elements and subsets we introduce the idea of a *point relation*. A point relation associated with an element $x \in X$ is a relation $x : I \rightharpoondown X$, defined by

$$(*, x') \in x \Leftrightarrow x' = x.$$

With x we denote both an element $x \in X$ and a relation $x : I \rightharpoondown X$. All such point relations are injective and because of their univalency even matchings. Point relations satisfy according to their definition the so-called point characteristics:

(PC1) $x \sqcap x' = id_{IX} \Leftrightarrow x = x'$ and
(PC2) for all relations $\rho : I \rightharpoondown X$ holds an identity $\rho = \sqcup_{x \sqsubseteq \rho} x$

The first characteristic states, that every element is represented by exactly one point relation, according to the second there is an one-to-one correspondence between a relation $\rho : I \rightharpoondown X$ and a subset S of X, whereby $S = \{x \in X | x \sqsubseteq \rho\}$. We use ρ for both a subset $\rho \subseteq X$ and a relation $\rho : I \rightharpoondown X$. The actual meaning can by seen from the context. The second characteristic also describes an inductive construction of boolean relation $\rho_I \rightharpoondown X$, what we will use for inductive proofs for claims about such relations.

Using these ideas we can formulate an algebraic characterization of the cardinality of relations:

Theorem 2.4: *A family of mappings* $|\cdot| : Rel(X, Y) \to \mathbb{N}$ *coincides with the cardinality of relations iff the following conditions are fulfilled:*

(a) $|\alpha| = 0 \Leftrightarrow \alpha = 0_{XY}$

(b) $|id_I| = 1$ *and* $|\alpha^\sharp| = |\alpha|$

(c) für α, β *mit* $\alpha \sqcap \beta = 0_{XY}$ *gilt* $|\alpha \sqcup \beta| = |\alpha| + |\beta|$

(d) (Dedekind inequality) If α *is univalent, then the inequalities* $|\beta \sqcap \alpha^\sharp \gamma| \leq |\alpha\beta \sqcap \gamma|$ *and* $|\alpha \sqcap \gamma\beta^\sharp| \leq |\alpha\beta \sqcap \gamma|$ *hold.*

The first three conditions look rather obviously; the forth is necessary to build a bridge between relations on I and other relations. Otherwise one could define a family of mappings, which has the above demanded properties on $Rel(I, I)$, but on other relations delivers the double value of the expected value. Such a family of mappings would fulfil the conditions (a)-(c), but it doesn't describe the cardinality of relations in common sense.

Proof: It is clear, that the cardinality of relations satisfies the properies above (the Dedekind inequality is already shown). Therefore we still need to show, that a familiy of mappings with the properties (a)-(d) describes the cardinality of relations.

First we notice, that for relation $\alpha : X \rightharpoondown Y$

$$(x, y) \in \alpha \Leftrightarrow x\alpha y^\sharp = id_I$$

obviously holds.

Let from now on $\alpha : X \rightharpoondown Y$, $\rho, \rho_0, \rho_1 : I \rightharpoondown X$ and $\mu : I \rightharpoondown Y$ be relations as well as $x : I \rightharpoondown X$ a point relation. Then hold the following claims (of which the last one will lead us directly to our desired goal):

(a) $|\alpha \sqcap x^\sharp\mu| = |x\alpha \sqcap \mu|$

(b) $|\alpha \sqcup \beta| + |\alpha \sqcap \beta| = |\alpha| + |\beta|$

(c) $|\alpha \sqcap (\rho_0 \sqcup \rho_1)^\sharp\mu| + |\alpha \sqcap (\rho_0 \sqcap \rho_1)^\sharp\mu| = |\alpha \sqcap \rho_0^\sharp\mu| + |\alpha \sqcap \rho_1^\sharp\mu|$

(d) $|x| = 1$

(e) $|\alpha \sqcap \rho^\sharp\mu| = \sum_{x \sqsubseteq \rho} |x\alpha \sqcap \mu| = \sum_{x \sqsubseteq \rho} \sum_{y \sqsubseteq \mu} |x\alpha y^\sharp|$

(f) $|\alpha| = \sum_{x \in X} \sum_{y \in Y} |x\alpha y^\sharp|$

Proof:

(a) Because for a point relation x both x and $x^\sharp$ are univalent, the following holds:

$$\begin{aligned}
|\alpha \sqcap x^\sharp\mu| &\leq |x\alpha \sqcap \mu| && \{ \text{ requirement 2.4(d), } x \text{ univalent } \} \\
&= |\mu \sqcap x\alpha| && \{ \ \alpha \sqcap \beta = \beta \sqcap \alpha \ \} \\
&\leq |x^\sharp\mu \sqcap \alpha| && \{ \text{ requirement 2.4(d), } x^\sharp \text{ univalent } \} \\
&= |\alpha \sqcap x^\sharp\mu| && \{ \ \alpha \sqcap \beta = \beta \sqcap \alpha \ \}
\end{aligned}$$

(b) First we calculate:

$$\begin{aligned}|\beta| &= |(\beta \sqcap \alpha^-) \sqcup (\beta \sqcap \alpha)| && \{ \text{ relational algebra } \} \\ &= |\beta \sqcap \alpha^-| + |\beta \sqcap \alpha| && \{ (\beta \sqcap \alpha^-) \sqcap (\beta \sqcap \alpha) = 0_{XY}, \text{ requ. 2.4(c) } \}\end{aligned}$$

and obtain after adding $|\alpha|$ on both sides

$$|\alpha| + |\beta \sqcap \alpha^-| + |\beta \sqcap \alpha| = |\alpha| + |\beta|$$

On the other hand holds

$$\begin{aligned}|\alpha \sqcup \beta| &= |\alpha \sqcup (\beta \sqcap \alpha^-)| && \{ \text{ relational algebra } \} \\ &= |\alpha| + |\beta \sqcap \alpha^-| && \{ \alpha \sqcap (\beta \sqcap \alpha^-) = 0_{XY}, \text{ requ. 2.4(c) } \}\end{aligned}$$

If we write this equation as

$$|\alpha| + |\beta \sqcap \alpha^-| = |\alpha \sqcup \beta|$$

and substitute with it the first two summands of the previous equation we get the claim.

This relationship is more intuitively for sets A and B known as

$$|A \cup B| = |A| + |B| - |A \cap B|$$

from elementary set theory; because of reasons, which will be soon clear we decided for a description without subtraction.

(c) Here it is enough to calculate straight forward and to use requirement 2.4(c):

$$\begin{aligned}|\alpha \sqcap (\rho_0 \sqcup \rho_1)^\sharp \mu| &= |\alpha \sqcap (\rho_0^\sharp \sqcup \rho_1^\sharp)\mu| && \{ (\alpha \sqcup \beta)^\sharp = \alpha^\sharp \sqcup \beta^\sharp \} \\ &= |\alpha \sqcap (\rho_0^\sharp \mu \sqcup \rho_1^\sharp \mu)| && \{ (\alpha \sqcup \beta)\gamma = \alpha\beta \sqcup \alpha\gamma \} \\ &= |(\alpha \sqcap \rho_0^\sharp \mu) \sqcup (\alpha \sqcap \rho_1^\sharp \mu)| && \{ \alpha \sqcap (\beta \sqcup \gamma) = (\alpha \sqcap \beta) \sqcup (\alpha \sqcap \gamma) \} \\ &= |\alpha \sqcap \rho_0^\sharp \mu| + |\alpha \sqcap \rho_1^\sharp \mu| - |\alpha \sqcap \rho_0^\sharp \mu \sqcap \rho_1^\sharp \mu| && \{ \text{ requ. 2.4(c) } \} \\ &= |\alpha \sqcap \rho_0^\sharp \mu| + |\alpha \sqcap \rho_1^\sharp \mu| - |\alpha \sqcap (\rho_0 \sqcap \rho_1)^\sharp \mu| && \{ \text{ see below } \}\end{aligned}$$

The Explanation for the step from the penultimate to the ultimate line is the following: as already mentioned, in general composition doesn't distribute over meet of relations. The two relations $\rho_0^\sharp \mu \sqcap \rho_1^\sharp \mu$ and $(\rho_0 \sqcap \rho_1)^\sharp \mu$ contain both exactly the pairs in $X \times Y$, of which the first entry is from $\rho_0 \cap \rho_1$ and the second from μ.

(d) Now the second part of requirement 2.4(d) has its entrance:

$$\begin{aligned}|x| &= |x \sqcap id_I x| && \{ x = id_I x \} \\ &= |x \sqcap id_I (x^\sharp)^\sharp| && \{ (\alpha^\sharp)^\sharp = \alpha \} \\ &\leq |x x^\sharp \sqcap id_I| && \{ \text{ requirement 2.4(d), } x \text{ univalent } \} \\ &= |id_I| && \{ \text{ clear } \} \\ &= 1 && \{ \text{ requirement 2.4(b) } \}\end{aligned}$$

On the other hand holds:

$$\begin{aligned}
1 &= |id_I| && \{ \text{ requirement 2.4(b) } \} \\
&= |id_I \sqcap xx^\sharp && \{ \ id_I = xx^\sharp \} \\
&\leq |id_I x \sqcap x| && \{ \text{ requirement 2.4(d), } id_I \text{ univalent } \} \\
&= |x| && \{ \ id_I x = x \ \}
\end{aligned}$$

So we obtained $|x| \leq 1 \leq |x|$, and the claim is shown.

(e) We show the first identity $|\alpha \sqcap \rho^\sharp \mu| = \sum_{x \sqsubseteq \rho} |x\alpha \sqcap \mu|$ via induction over the number of point relations contained in ρ:

Induction beginning: in the case of $\rho = 0_{IX}$ both sides become zero: the left side because of requirement 2.4(a), and the right side consists only of an empty sum. If ρ contains exactly one element, then ρ is a point relation and the claim is reduced to part (b).

Induction step: Let $\rho = \rho_0 \sqcup x$ and the claim for ρ_0 can be assumed as already proofed by induction hypothesis. Furthermore we can demand, that $\rho_0 \sqcap x = 0_{IX}$. It is important, that all relations $\rho : I \rightharpoonup X$ can be constructed in such manner because of PC(2). Now we can conclude as follows:

$$\begin{aligned}
|\alpha \sqcap \rho^\sharp \mu| &= |\alpha \sqcap (\rho_0 \sqcup x)^\sharp \mu| \\
&= |\alpha \sqcap \rho_0^\sharp \mu| + |\alpha \sqcap x^\sharp \mu| - |\alpha \sqcap (\rho_0 \sqcap x)^\sharp \mu| && \{ \text{ requ. (c) } \} \\
&= |\alpha \sqcap \rho_0^\sharp \mu| + |\alpha \sqcap x^\sharp \mu| && \{ \ \rho_0 \sqcap x = 0_{IX} \ \} \\
&= \textstyle\sum_{x \sqsubseteq \rho_0} |x\alpha \sqcap \mu| + |\alpha \sqcap x^\sharp \mu| && \{ \text{ induction hypothesis } \} \\
&= \textstyle\sum_{x \sqsubseteq \rho_0} |x\alpha \sqcap \mu| + |x\alpha \sqcap \mu| && \{ \text{ part (a) } \} \\
&= \textstyle\sum_{x \sqsubseteq \rho} |x\alpha \sqcap \mu| && \{ \ \rho_0 \sqcap x = 0_{IX} \ \}
\end{aligned}$$

The second identity is shown analogously by induction over the construction of μ, we will take a closer look only at the induction beginning.

In this case is to show $|x\alpha \sqcap \mu| = |x\alpha\mu^\sharp|$ for empty μ and an arbitrary point relation μ. For $\mu = 0_{IY}$ both sides are zero because of requirement 2.4(a). So it remains to show, that for point relations $y : I \rightharpoonup Y$ the equality $|x\alpha \sqcap y| = |x\alpha y^\sharp|$ holds. If $(x, y) \notin \alpha$, then $xa \sqcap y$ results in the empty relation 0_{IY} and $xay^\sharp$ becomes 0_{II}; both sides are zero according to requirement 2.4(a). The more interesting case is, if $(x, y) \in \alpha$: $xa \sqcap y$ describes the point relation y, and $xay^\sharp$ represents id_Y. Here both sides become one because of part (c) and requirement 2.4(b).

(f) The claim follows from part (d) by choosing the universal relations ∇_{IX} resp. ∇_{IY} for ρ and μ. Note, that $\alpha = \alpha \sqcap \nabla_{XY} = \alpha \sqcap \nabla_{XI} \nabla_{IY} = \alpha \sqcap \nabla_{IX}^\sharp \nabla_{IY}$ holds. More important is the interpretation of this result: the expressions $x\alpha y^\sharp$ are one, if $(x, y) \in \alpha$, and they are zero, if $(x, y) \notin \alpha$. Because the summation extends over all elements from X and Y it describes exactly the number of pairs $(x, y) \in X \times Y$, which are also contained in α, i.e., the cardinality of α. This completes the proof. ■

So far we followed [Kaw] closely, now we will generalize the cardinality of relations by extending the range of the cardinality from the natural numbers to an arbitrary commutative cancellative monoid $(M, +, 0)$ with undivisible zero (see also the chapter about monoids and semirings). On such a monoid an order $\leq_M$ is defined by $x \leq_M y \Leftrightarrow \exists\, a : x + a = y$
The fundamental properties of a cardinality function are the requirements of theorem 2.4, and they suffice to determine a cardinality function in our case of a generalized range uniquely:

Theorem 2.5: *A family of mappings $|\cdot|$ from $Rel(X, Y)$ into $(M, +, 0)$, where $(M, +, 0)$ is a commutative monoid with undivisible zero, is uniquely determined by the following requirements:*

(a) $|\alpha|_M = 0 \Leftrightarrow \alpha = 0_{XY}$

(b) $|id_I|_M = E$*, where* $E \in M\backslash\{0\}$ *is choosen arbitrarily, but constant, and* $|\alpha^\sharp|_M = |\alpha|_M$

(c) for α, β *mit* $\alpha \sqcap \beta = 0_{XY}$ *holds* $|\alpha \sqcup \beta|_M = |\alpha|_M + |\beta|_M$

(d) (Dedekind inequality) If α *is univalent, then hold the inequalities* $|\beta \sqcap \alpha^\sharp\gamma|_M \leq_M |\alpha\beta \sqcap \gamma|_M$ *and* $|\alpha \sqcap \gamma\beta^\sharp|_M \leq_M |\alpha\beta \sqcap \gamma|_M$

Furthermore holds $|\alpha| = n \Leftrightarrow |\alpha|_M = n * E$*, where where* $n * x$ *denotes the* n*-time summation of* E*.*

Proof: For the proof we don't need a lot, we did the most necessary already in the proof of theorem 2.4. What we need is the following

Help claim: *In a cancellative monoid $(M, +, 0)$ with undivisable zero and order $\leq_M$ follows from $a \leq b \leq a$ that $a = b$, i.e., $\leq_M$ is antisymmetric.*

Proof: Let $a, b \in M$ be arbitrary. Then holds:

$a \leq_M b \leq_M a \Rightarrow$
{ definiton of $\leq_M$ }
$\exists\, x, y : a + x = b \wedge b + y = a \Rightarrow$
{ set }
$\exists\, x, y : a + x + y = a \Rightarrow$
{ cancellativity }
$x + y = 0 \Rightarrow$
{ undivisibility of zero }
$x = y = 0 \Rightarrow$
{ set }
$a = b$ ■

We conclude now analogously to the proof of theorem 2.4, with the difference, that we replace $|\cdot|$ by $|\cdot|_M$ and $\leq$ by $\leq_M$. The proof of the parts (b), (c), (e) and (f) taken over without changes, because as algebraic operations they use only addition and equality (this is the reason, for what we decided to choose a representation without minus in part (c): in a monoid a inverse is in general not defined). In the proofs of part (a) and (d) we used constructs of the form $a \leq b \leq a \Rightarrow a = b$, which hold here also due to our help claim. The proof of part (d) in the new context yields here $|x| \leq_M E \leq_M |x|$ and according to our help claim $|x| = E$. The last statement of theorem 2.5 follows from the bijectivity of the mapping $P_E : \mathbb{N}_0 \to \{E^n | n \in \mathbb{N}_0\}$ with $P_E(n) = E^n$ for $E \neq 0$, shown in the chapter about monoids and semirings. ■

2.5 Tests on Relations

A closely related idea of the point relations are the so-called test relations or shortly tests. A test relation τ on a set X is an endorelation on X with the property $\tau \sqsubseteq id_X$. Obviously exists for each subset $T \subseteq X$ of X an associated test relation τ on X, characterised by $(x, y) \in \tau \Leftrightarrow x = y \wedge x \in T$. Therefore the test relations on a set X are in a one-to-one correspondence with the relations from I into X.

For a subset $S \subseteq X$ we denote the test relation belonging to S by $\tau(S)$; we use $\tau(x)$ as an abbreviation for $\tau(\{x\})$ in the case of a singleton set $\{x\}$. In the same way we write the test relation belonging to a relation $\rho : I \rightharpoondown X$ as $\tau(\rho)$. With these writings it is clear, that the following equality holds:

$$\tau(\rho)\nabla_{XY} = \rho^\sharp \nabla_{IY}$$

Because a test τ as a subrelation of the identity satisfies the property $\tau = \tau^\sharp$, we can show the identity $|\tau(\rho)\alpha| = |\alpha^\sharp \sqcap \rho^\sharp \nabla_{IY}|$ for arbitrary relations $\alpha : X \rightharpoondown Y$:

$|\tau(\rho)\alpha| =$
  { neutrality of ∇_{XY} concerning meet }
$|\tau(\rho)\alpha \sqcap \nabla_{XY}| =$
  { $\tau(\rho)$ matching, corollary 2.2(b) }
$|\alpha^\sharp \sqcap (\tau(\rho))^\sharp \nabla_{XY}| =$
  { $(\tau(\rho))^\sharp = \tau(\rho)$ }
  $|\alpha^\sharp \sqcap \tau(\rho)\nabla_{XY}| =$
{ $\tau(\rho)\nabla_{XY} = \rho^\sharp \nabla_{IY}$ }
  $|\alpha^\sharp \sqcap \rho^\sharp \nabla_{IY}|$

Easy to see is the property

$$\rho\alpha = \rho\tau(\rho)\alpha$$

for arbitrary relations $\rho : I \rightharpoondown X$ and $\alpha : X \rightharpoondown Y$.
For point relations $x : I \rightharpoondown X$ holds the identity

$$|x\alpha| = |\tau(x)\alpha|$$

because of $(*, y) \in x\alpha \Leftrightarrow (x, y) \in \tau(x)\alpha$. For general relations holds however only the inequality

$$|\rho\alpha| \leq |\tau(\rho)\alpha|$$

The argument is, that $|\rho\alpha|$ counts the elements reachable from ρ under α, whereas $|\tau(\rho)\alpha$ is the number of all from relation pairs of α emerging from ρ.

A boolean relation $\rho : I \rightharpoondown X$ associated with the test relation $\tau(\rho)$ can be written as

$$\tau = \sqcup_{x \sqsubseteq \rho} x^\sharp x = \sqcup_{x \sqsubseteq \rho} \tau(x)$$

where x is always a point relation. This property we will often exploit for conducting proofs about test relation via induction over their construction. Note that every test relation can be written in the way above.

3 semirings

The following chapter will lead us in completely other regions, which have at a first look nothing in common with relations. But we will use semirings as a valuable aid in the further course.

A good introduction in the theory of semirings offer the chapters four and five from [Ml], from where we took also the proofs of the properties of tests.

3.1 Monoids and Semirings

Definition (Monoid): A *monoid* is a triple $(M, \circ, e)$, where $\circ$ is an associative mapping from $M \times M$ into M with neutral element e, i.e.,

$$m \circ e = e \circ m = m$$

holds for all $m \in M$.

The operation $\circ$ we write often as $+$ or $\cdot$; we then speak of an *additive* resp. *multiplicative* monoid.

We define the powers in a monoid inductively by $x^0 = e$ and $x^{n+1} = x \circ x^n$. In an additive monoid we often write $n * x$ instead of x^n.

If the operation $\circ$ is additionally commutative, we call the monoid also *commutative.*

A monoid is called *left-cancellative*, if from $a \circ x = a \circ y$ follows $x = y$. If from $x \circ a = y \circ a$ follows, that $x = y$, then the monoid is called *right-cancellative.* A both left- and right-cancellative monoid is simply called *cancellative.*

In an additive monoid the neutral element is called *zero*, in a multiplicative one it has the name *one.*

The neutral element e is called *indivisible*, if $a \circ b = e$ implies $a = b = e$.

In a cancellative monoid with undivisible neutral element e holds the equivalence

$$x^m \neq x^n \Leftrightarrow x \neq e$$

This can be easily seen by proving the contraposition

$$x^m = x^n \Leftrightarrow x = e$$

The implication from the left to the right is trivial. To show the other implication assume w.l.o.g., that $m > n$, from which because of the cancellativity follows $x^{m-n} = e$ and hence according to the indivisibility of e the equality $x = e$.

An example for a non-commutative monoid are the words over an alphabet with two or more elements with respect to the concatenation as operation and the empty word as neutral element. This monoid is even cancellative. The relations over a set X form even a commutative monoid with the join as operation and 0_{XX} as neutral element, but no cancellative one. If one replaces the join as operation by the composition and the chooses id_X as neutral element, one gets again a monoid, but this time neither cancellative nor commutative.

Definition (Semiring): A quintupel $(M, +, \cdot, 0, 1)$ is called a *semiring* if the following properties are satisfied:

- $(M, +, 0)$ is a commutative monoid.
- $(M, \cdot, 1)$ is a monoid
- 0 is a annihilator with respect to. $\cdot$, i.e.,

$$m \cdot 0 = 0 \cdot m = 0 \ \forall m \in M$$

- $\cdot$ distributes over $+$, i.e.,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c),\ (a + b) \cdot c = (a \cdot c) + (b \cdot c)\ \forall a, b, c \in M$$

We call the operation $+$ *addition*; $\cdot$ has the name *multiplication*. Like in real life the operator $\cdot$ binds stronger than $+$

3.2 Order, Infimum and Supremum

A semiring is called *idempotent*, if its addition is idempotent, i.e., $a+a = a$ holds for all elements a of this semiring. On such an idempotent semiring a natural order is given by

$$a \sqsubseteq b \Leftrightarrow a + b = b$$

For the proof of the order properties see [Ml]. An other property immediately following from the definition is $x \sqsubseteq x + y$ for all semiring elements x and y.

Because of $0+x = x$ for all x zero is the least element with respect to this order. Furthermore, for two elements x and y the sum $x + y$ is the least upper bound, i.e., the supremum, characterised by $x + y \sqsubseteq z \Leftrightarrow x \sqsubseteq z \wedge y \sqsubseteq z$. For details see here also [Ml].

Addition and multiplication preserve inequalities: for all elements x, y and z hold the implications $x \sqsubseteq y \Rightarrow x + z \sqsubseteq y + z$ and $x \sqsubseteq z \Rightarrow x \cdot z \sqsubseteq y \cdot z$, analogously for addition and multiplication from the left with z. The proofs are rather simple, first we give the one for addition (requirement id here $x \sqsubseteq y$):

$$\begin{aligned} x + z &\sqsubseteq x + z + y + z && \{\ a \sqsubseteq a + b\ \} \\ &= x + y + z && \{\ \text{commutativity and idempotency of } +\ \} \\ &= y + z && \{\ x \sqsubseteq y \Rightarrow x + y = y\ \} \end{aligned}$$

Similarly we can argue for the multiplication:

$$\begin{aligned} x \cdot z &\sqsubseteq x \cdot z + y \cdot z && \{\ a \sqsubseteq a + b\ \} \\ &= (x + y) \cdot z && \{\ \text{distributivity}\ \} \\ &= y \cdot z && \{\ x \sqsubseteq y \Rightarrow x + y = y\ \} \end{aligned}$$

By contrast to the supremum the existence of the infimum, the greatest lower bound of two elements x and y is not guaranteed in general. In the case of its existence we denote it by $inf(x, y)$. It is characterised symmetrically to the supremum by $z \sqsubseteq inf(x, y) \Leftrightarrow z \sqsubseteq x \wedge z \sqsubseteq y$.

We see now, that the relations over a set X with the join as addition and the composition as multiplication form an idempotent semiring. The former introduced order $\sqsubseteq$ coincides with the order induced by teh join as addition. The meet of relations fits perfectly in the picture of a semiring: it corresponds to the infimum of two relations.

3.3 Tests on Semirings

An important group of elements in an idempotent with natural order $\sqsubseteq$ semiring is formed by the so-called *tests*, which are characterised as follows:

Definition (Tests in semirings): An element p of an idempotent semiring with natural order $\sqsubseteq$ is called *test*, if $p \sqsubseteq 1$ holds and if exists an element $\neg p$, the so-called *complement* of p, with the properties $p + \neg p = 1$ and $p \cdot \neg p = 0 = \neg p \cdot p$.

Because we will use tests intensively we summarize some important properties of tests:

Theorem 3.1 (Properties of tests):

1. For a test p $\neg p$ is also a test, and $\neg(\neg p) = p$ holds.
2. 0 and 1 are tests, and it holds $\neg 0 = 1$ and $\neg 1 = 0$
3. The set of tests is closed under addition and multiplication, and the de-Morgan laws hold: $\neg(p + q) = \neg p \cdot \neg q$ and $\neg(p \cdot q) = \neg p + \neg q$
4. Tests are idempotent with respect to multiplication, i.e., for a test p holds $p \cdot p = p$.
5. On tests multiplication and infimum coincide, i.e., for tests p, q and r holds $r \sqsubseteq p \cdot q \Leftrightarrow r \sqsubseteq p \wedge r \sqsubseteq q$.

6. Multiplication on tests is commutative: for tests p and q holds $p \cdot q = q \cdot p$.

The proofs can be found in [Möl]

Now we apply these tools on relations. First we want to determine the tests in $Rel(X, X)$. We know already, that the natural order in $Rel(X, X)$ corresponds to the relation $\sqsubseteq$ introduced in the chapter about relations. Because the one in $Rel(X, X)$ is id_X a test has in this case to be a relation τ with $\tau \sqsubseteq id_X$. To see that every relation τ with $\tau \sqsubseteq id_X$ is a test we write τ as $\tau(\rho)$ for a suitable subset $\rho \subseteq X$ and note, that $\tau(X \backslash \rho)$ is the complement of τ.

The infimum of two test relations τ_1 and τ_2 is obviously also a test relation, so we know immediately, that $\tau_1 \sqcap \tau_2 = \tau_1 \tau_2$ holds (note, that $\sqcap$ is the infimum operator on relations). From the commutativity of the multiplication on test we obtain $\tau_1 \tau_2 = \tau_2 \tau_1$, and the idempotency of the multiplication on tests delivers $\tau\tau = \tau$ for arbitrary test relations τ.

3.4 Matrices over Semirings

Semirings are mathematically already rather mighty constructions; one has not so much possibilities like on the real numbers, but it makes sense to deal with matrices over semirings:

Definition (Matric operations over semirings): Let $(M, +, \cdot, 0, 1)$ be a semiring. For two $m \times n$-matrices X and Y with entries from M we define the *sum* $Z := X + Y$ of X and Y by $Z_{i,j} = X_{i,j} + Y_{i,j}$. The *product* $Z := X \cdot Y$ of a $m \times k$-matric X and a $k \times n$-matric Y is a $m \times n$-matric, defined by $Z_{i,j} = \sum_{l=1}^{k} X_{i,l} \cdot Y_{l,j}$.

The set of all $m \times n$-matrices with entries from a set M we denote with $M(m \times n)$. For semirings $S = (M, +, \cdot, 0, 1)$ and natural numbers $n > 0$ we introduce square matrices $N_{S,n}$ and $E_{S,n} \in M(n \times n)$, charakterised by $[N_{S,n}]_{i,j} = 0 \forall i, j$ and $[E_{S,n}]_{i,j} = \delta_{i,j}$ ($E_{S,n}$ is a diagonal matric with ones on the main diagonal and zeros on the remaining entries). Now we can extend the semiring properties on matrices:

Lemma 3.2 (Matrice semirings): Let $S = (M, +, \cdot, 0, 1)$ be a semiring. Then for every $n \in \mathbb{N}^+$ $(M(n \times n), +, \cdot, N_{S,n}, E_{S,n})$ with the operations $+$ and $\cdot$ introduced above is also a semiring.

The proof can be found in every book about linear algebra and is here omitted.

4 Fuzzy Relations

4.1 Introduction

Fuzzy relations are, as already mentioned, "weighted" relations, where the "weights" have values between zero and one. To define operations on fuzzy relations we first introduce some operations on the unit interval $[0, 1]$

We use the four binary operators $\wedge$, $\vee$, $\ominus$ and $\oplus : [0,1] \times [0,1 \to [0,1]$, defined by

- $a \vee b =$max$\{a, b\}$
- $a \wedge b =$min$\{a, b\}$
- $a \ominus b =$max$\{0, a - b\}$
- $a \oplus b =$min$\{1, a + b\}$

If we imagine the boolean values `true` and `false` as corresponding to one resp. zero, the operators $\wedge$ and $\vee$ are a natural generalisation of the common `or`- and `and`-operations. $\ominus$ and $\oplus$ correspond to the common subtraction resp. addition, cut if at one resp. zero. We introduce an unary operator $\cdot^\bullet$, defined by $a^{bullet} = 0$ if $a = 0$ and $a^\bullet = 1$ otherwise. A real number is called *boolean*, if it is zero or one. The following properties are easy to see and are given without proof:

(a) $a \leq a^\bullet = a^{\bullet\bullet}$ and $(a \ominus b) \wedge (b \ominus a) = 0$

(b) $a \ominus b = (a \vee b) - b$, particularly $a \ominus b = a - b$ if $b \leq a$

(c) $a = (a \ominus b) \oplus (a \wedge b)$

(d) If $a \leq c$ and $b \leq c \ominus a$, then $a \oplus b = a + b \leq c$

(e) If a is boolean, then $a \wedge (b \ominus c) = (a \wedge b) \ominus (a \wedge c)$

(f) If a is boolean, then $a \wedge (b \oplus c) = (a \wedge b) \oplus (a \wedge c)$

A *fuzzy relation* α from a set X into a set Y, written as $\alpha : X \rightharpoondown Y$ is a mapping $\alpha : X \times Y \to [0,1]$. Note that a fuzzy relation is a total function, it is defined on all pairs from $X \times Y$, contrary to a old fashioned relation, which in general doesn't contain all pairs of $X \times Y$. Pairs $(x, y) \in X \times Y$ with $\alpha(x, y) > 0$ we call *edges* of α; consequently

$$|\{(x, y) \in X \times Y : \alpha(x, y) > 0\}|$$

is the *number of edges* of α.

For fuzzy relations $\alpha, \beta : X \rightharpoondown Y$ the relations $\alpha \ominus \beta$,$\alpha \oplus \beta$,$\alpha^\bullet : X \rightharpoondown Y$ are defined pontwise, i.e.:

- $\forall x \in X \ \forall y \in Y : (\alpha \ominus \beta)(x, y,) = \alpha(x, y) \ominus \beta(x, y)$
- $\forall x \in X \ \forall y \in Y : (\alpha \oplus \beta)(x, y,) = \alpha(x, y) \oplus \beta(x, y)$
- $\forall x \in X \ \forall y \in Y : \alpha^\bullet(x, y) = (\alpha(x, y))^\bullet$

The already on the old relations defined operators $\sqcap$, $\sqcup$ and $^\sharp$ become overloaded and have a new look:

- $\forall x \in X \ \forall y \in Y : (\alpha \sqcup \beta)(x, y,) = \alpha(x, y) \vee \beta(x, y)$

- $\forall x \in X\ \forall y \in Y : (\alpha \sqcap \beta)(x, y,) = \alpha(x, y) \wedge \beta(x, y)$
- $\forall x \in X\ \forall y \in Y : \alpha^\sharp(x, y) = \alpha(y, x)$

In the case of boolean relations the definitions coincide.

The universal relation ∇_{XY} and the empty relation 0_{XY} have the following characterisations:

- $\forall x \in X\ \forall y \in Y : \nabla_{XY}(x, y) = 1$
- $\forall x \in X\ \forall y \in Y : 0_{XY}(x, y) = 0$

The identity relation id_X has the property $id_X(x, x) = 1$ and $id_X(x, y) = 0$ if $x \neq y$. For two fuzzy relations $\alpha, \beta : X \rightharpoondown Y$ holds $\alpha \sqsubseteq \beta$, if $\alpha(x, y) \leq \beta(x, y)$ holds for all $(x, y) \in X \times Y$. We call a fuzzy relation *univalent*, if $\alpha^\sharp\alpha \sqsubseteq id_Y$ holds. Note that in the case of boolean relations these new definitions are compatible with the old ones.

In a slightly different way, but in the case of boolean relations still compatible with usual relations, the composition of fuzzy relations is defined:
Let $\alpha : X \rightharpoondown Y$ and $\beta : Y \rightharpoondown Z$ be fuzzy relations. Their composition $\alpha\beta$ is a fuzzy relation $\alpha\beta : X \rightharpoondown Z$, given by $\alpha\beta(x, z) = \bigvee_{y \in Y}(\alpha(x, y) \wedge \beta(y, z))$.

If one imagines α and β as a system of pipes with a maximal throughput given by the values of α resp. β the value of $\alpha\beta(x, z)$ corresponds to the maximal amount, what can be sent from x to z via one pipe of α combined with one pipe of β. Here we see the first connections with network flows.

The following properties of fuzzy relations are trivial and easy to prove:

Propostion 4.1: *Let $\alpha, \beta, \gamma : X \rightharpoondown Y$ and $\mu : V \rightharpoondown X$ be fuzzy relations. Then holds:*

(a) $\alpha \ominus \beta \sqsubseteq \alpha$, $\alpha \ominus 0_{XY} = \alpha$ *and* $(\alpha \ominus \beta) \sqcap (\beta \ominus \alpha) = 0_{XY}$

(b) If $\alpha \sqsubseteq \gamma$ *and* $\beta \sqsubseteq \gamma \ominus \alpha$, *then* $\alpha \oplus \beta \sqsubseteq \gamma$.

(c) If γ *is boolean, then* $(\alpha \ominus \beta) \sqcap \gamma = (\alpha \sqcap \gamma) \ominus (\beta \sqcap \gamma)$ *and* $(\alpha \oplus \beta) \sqcap \gamma = (\alpha \sqcap \gamma) \oplus (\beta \sqcap \gamma)$.

(d) If μ *is boolean and univalent, then* $\mu(\alpha \ominus \beta) = (\mu\alpha) \ominus (\mu\beta)$ *and* $\mu(\alpha \oplus \beta) = (\mu\alpha) \oplus (\mu\beta)$ *and symmetrically* $(\alpha \ominus \beta)\mu = \alpha\mu \ominus \beta\mu$ *and* $(\alpha \oplus \beta)\mu = \alpha\mu \oplus \beta\mu$.

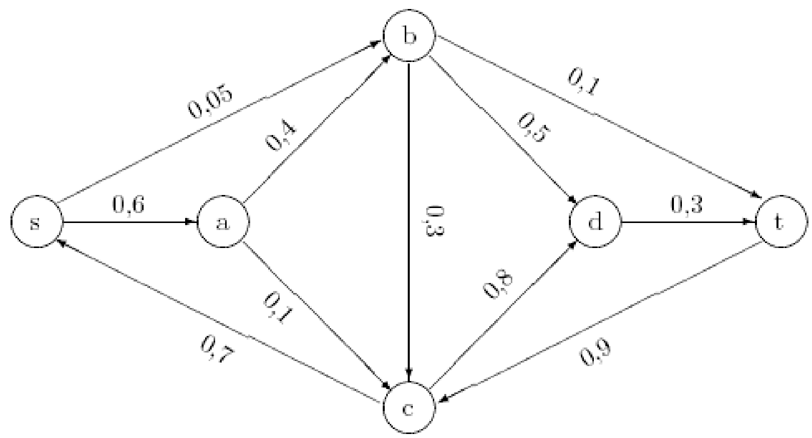
(e) $\alpha \sqsubseteq \alpha^\bullet = \alpha^{\bullet\bullet}$, ${0_{XY}}^\bullet = 0_{XY}$, ${\nabla_{XY}}^\bullet = \nabla_{XY}$ *and* ${id_X}^\bullet = id_X$.

(f) $\alpha^{\sharp\bullet} = \alpha^{\bullet\sharp}$, $(\alpha \sqcap \beta)^\bullet = \alpha^\bullet \sqcap \beta^\bullet$, $(\alpha \sqcup \beta^\bullet = \alpha^\bullet \sqcup \beta^\bullet$ *and* $(\alpha\beta)^\bullet = \alpha^\bullet\beta^\bullet$.

(g) $(\alpha \ominus \beta)^\sharp = \alpha^\sharp \ominus \beta^\sharp$

4.2 Inuition of Fuzzy Relations

After this dry algebraic matter we will consider, how to imagine fuzzy relations in an intuitive way. Of special interest will be fuzzy endorelations $\alpha : X \rightharpoondown X$. Such fuzzy relations we will imagine as a directed graph: every element of X corresponds to a node oh the graph, and every edge of the graph is labelled with the corresponding value of α. A further convention, to keep the pictures clear, will be, that edges with a value of zero are omitted in the depiction.



The fuzzy relation depicted above will accompany us till the end, therefore it deserves a closer look: it is a fuzzy endorelation on the set {a,b,c,d,s,t} with e.g. α(a,b)=0.4 and α(t,c)=0.9. From s to d or from c to a no edges are depicted, that means α(s,d) and $\alpha(c, a)$ are both zero. It is possible, that between two nodes exists a pair of antiparallel edges; the reason, that it is not the case in our example relation, is, that we will later deal with a class of fuzzy endorelations, which doesn't allow more than one edge between to nodes.

4.3 Fuzzy Relations and Semirings

IN this chapter we use our tools from the chapter about semirings. One sees easily, that $([0.1], \vee, \wedge, 0, 1)$ is an idempotent semiring. Important is now, that the matric semiring induced by it is in close connection with fuzzy relations. Because we restrict ourselves to finite sets we can numerate the elements of the sets X and Y, which take part in a fuzzy relation $\alpha : X \rightharpoondown Y$, i.e., we write X as $\{x_1, x_2, \vdots, x_m\}$ and Y as $\{y_1, y_2, \vdots, y_n\}$. A fuzzy relation α is identified with an $m \times n$-matric A with entries from [0,1], given by $A_{i,j} = \alpha_{i,j}$.

The $\sqsubseteq$-relation was defined by componentwise comparison, now we can rely on the matrices A and B belonging to two fuzzy relations α and β and write $A \sqsubseteq B$ iff $A + B = B$.

Matrices produced by boolean relations can be recognized by containing only zeros and ones as entries, particularly point relations $I \rightharpoondown X$ have the form

$(0, 0, \therefore, 1, 0, \therefore, 0)$, where the one stays on the position given by the number of the corresponding element in the chosen numeration of X. Test relations are incomplete diagonal matrices; they have on the main diagonal either ones or zeros and on the remaining positions everywhere zeros, while the matric corresponding to the identity is exactly $E_{S,n}$.

The pendant of the converse in the matrices' world is the transposition of matrices. The composition $\alpha\beta$ of two fuzzy relations corresponds to the product of the associated matrices. The composition of fuzzy relations is associative just as its algebraic form, the multiplication of matrices over a semiring.

4.4 Tests on Fuzzy Relations

As they are a idempotent semiring fuzzy relations can call tests their own. We will now determine and investigate tests on fuzzy relations.

Because tests are always lower or equal to one and the one in our case is a fuzzy relation described by a matric with ones on the main diagonal and zeros on the other positions tests have to be also diagonal matrices. We can strengthen this and show, that on the main diagonal an appear only zeros and ones: assumed on the main diagonal of hypothetical matric P associated with a test fuzzy relation a entry $P_{ii} \in]0, 1[$ exists. Then in the complement $\neg P$ of P the entry at position (i, i) had to be one, so that $(P + \neg P)_{ii}$ becomes one (this has to be because of the requirement $P + \neg P = 1$ on tests). But then the product $P \cdot \neg P$ contains at the position (i, i) the value $P_{i,i} \neq 0$, a contradiction to the test property $P \cdot \neg P = 0$. The fact, that all diagonal matrices with entries either zero or one are tests, can be shown, if one sees, that one can obtain the complement of such a matric by swapping ones and zeros on the main diagonal and not changing the remaining zeros.

Because we will often work with the complement of test fuzzy relations we introduce the easier readable writing τ^c for the complement $\neg\tau$.

Furthermore the connections between tests $\tau : X \rightharpoondown X$, point relations $\rho : I \rightharpoondown X$ and fuzzy relations $\alpha : X \rightharpoondown Y$ hold as traditionally:

$$\tau(\rho)\nabla_{XY} = \rho^\sharp \nabla_{IY},$$

$$\rho\alpha = \rho\tau(\rho)\alpha$$

and

$$\alpha \sqcap \rho^\sharp \rho^- = \tau(\rho)\alpha\tau(\rho^-) = \tau(\rho)\alpha(\tau(\rho))^c$$

5 Cardinality of Fuzzy Relations

5.1 Definition and fundamental properties

Let from now on X, Y and Z be finite sets.

Definition (Cardinality of Fuzzy Relations): The cardinality $|\alpha|$ of a fuzzy relation $\alpha : X \rightharpoondown Y$ is given by

$$|\alpha| = \sum_{x\in X} \sum_{y\in Y} \alpha(x,y)$$

Obviously the cardinality of a fuzzy relation is nonnegative real number. It fulfils the following properties: **Proposition 5.1:** *Let $\alpha, \beta, \gamma : X \rightharpoondown Y$ be fuzzy relations and $x \in X$ and $y \in Y$. Then holds:*

(a) $|x\alpha y^\sharp| = \alpha(x,y)$ and $x\alpha y^\sharp = \alpha(x,y) \cdot id_I$.

(b) $|\alpha \ominus \beta| = |\alpha \sqcup \beta| - |\beta|$. If $\beta \sqsubseteq \alpha$, then holds particularly $|\alpha \ominus \beta| = |\alpha| - |\beta|$.

(c) If $\alpha \sqsubseteq \gamma$ and $\beta \sqsubseteq \gamma \ominus \alpha$, then holds $|\alpha \oplus \beta| = |\alpha| + |\beta|$.

The properties (b) and (c) are easy to see; we will concentrate, because it is important for the following, on property (a).

x and y are fuzzy relations from I into X resp. Y. We describe these fuzzy relations and the fuzzy relation α - as explained in the chapter about semirings - as matrices. First we numerate the elements of X and Y in such a way, that x is mapped onto the number $l(x)$ and y onto $l(y)$. The matrices associated with the point relations x and y become line vectors of the length $m := |X|$ and $n := |Y|$, which contain everywhere zeros except on the position $l(x)$ resp. $l(y)$, on which they have a single one. These two vectors we call e_x and e_y. α is represented by a $m \times n$-matric, which contains at the position (i,j) the value $\alpha(i,j,)$ how it is prescribed by the chosen numeration of X and Y. If we remember, that the converse $\cdot^\sharp$ of relations corresponds with the transposition of matrices we can determine the matric associated with $x\alpha y^\sharp$ as follows:

$$e_x \cdot \begin{pmatrix} \alpha(1,1) & \alpha(1,2) & \dots & \alpha(1,n) \\ \alpha(2,1) & \alpha(2,2) & \dots & \alpha(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m,1) & \alpha(m,2) & \dots & \alpha(m,n) \end{pmatrix} \cdot e_y^t =$$

$$\begin{pmatrix} 0, & 0, & \dots & ,1, & 0, & \dots & ,0 \end{pmatrix} \cdot \begin{pmatrix} \alpha(1,1) & \alpha(1,2) & \dots & \alpha(1,n) \\ \alpha(2,1) & \alpha(2,2) & \dots & \alpha(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m,1) & \alpha(m,2) & \dots & \alpha(m,n) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} \alpha(x,1) & \alpha(x,2), & \dots, & \alpha(x,n) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} \alpha(x,y) \end{pmatrix}$$

Note that the matrices contain real numbers as entries, but that the operation $|\cdot|$ is defined as matric multiplication of matrices over the semiring $([0,1], \vee, \wedge, 0, 1)$.

The result is a 1×1-matric as we could have expected, because $x\alpha y^\sharp$ is a fuzzy endorelation on the singleton set I. So we have showed $x\alpha y^\sharp = \alpha(x,y) \cdot id_I$; part (a) is then clear, because according to the definition of the cardinality of fuzzy relations one has to summarise only over the pair $(*,*)$. ■

5.2 The Dedekind Inequality for Fuzzy Relations

It is fascinating, that the Dedekind inequality and its consequences hold also for fuzzy relations:

Theorem 5.2 (Dedekind inequality for fuzzy relations): *Let $\alpha : X \rightharpoondown Y, \beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ be fuzzy relations. If α is univalent, i.e., $\alpha^\sharp\alpha \sqsubseteq id_Y$, then the following inequalities hold:*

$$|\alpha \sqcap \gamma\beta^\sharp| \leq |\alpha\beta \sqcap \gamma| \qquad \textit{and}$$
$$|\beta \sqcap \alpha^\sharp\gamma| \leq |\alpha\beta \sqcap \gamma|.$$

Proof: Because α is univalent $\alpha^\sharp\alpha \sqsubseteq id_Y$ holds or in detail:

$$(\alpha^\sharp\alpha)(y,y') = \bigvee_{x\in X}(\alpha(x,y) \wedge \alpha(x,y')) \leq id_Y(y,y')$$

Assume now it exists an $x \in X$, so that there are two distinct elements $y, y' \in Y$ with $\alpha(x,y) > 0$ and $\alpha(x,y') > 0$. Then $(\alpha\sharp\alpha)(y,y'))$ would be greater than zero, a contradiction to $\alpha^\sharp\alpha \sqsubseteq id_Y$. That means, for each $x \in X$ at most one $y \in Y$ exists with $\alpha(x,y) > 0$. Therefore holds:

$$\begin{aligned} |\alpha\beta \sqcap \gamma| &= \textstyle\sum_{x\in X, z\in Z} \bigvee_{y\in Y}(\alpha(x,y) \wedge \beta(y,z) \wedge \gamma(x,z)) \\ &= \textstyle\sum_{x\in X, y\in Y, z\in Z}(\alpha(x,y) \wedge \beta(x,y) \wedge \gamma(x,z)) \end{aligned}$$

because the disjunction delivers at most one time a value unequal to zero and otherwise only zero. Because of the inequality $max(S) \leq \sum_{s\in S} s$ for all $S \subseteq \mathbb{R}_0^+$

we have $|\alpha\beta \sqcap \gamma| \leq \sum_{x\in X, y\in Y, z\in Z}(\alpha(x,y) \wedge \beta(y,z) \wedge \gamma(x,z))$ and can now conclude:

$$\begin{aligned}|\alpha \sqcap \gamma\beta^\sharp| &\leq \textstyle\sum_{x\in X, y\in Y, z\in Z}(\alpha(x,y) \wedge \gamma(x,z) \wedge \beta^\sharp(z,y)) \\ &= \textstyle\sum_{x\in X, y\in Y, z\in Z}(\alpha(x,y) \wedge \beta(y,z) \wedge \gamma(x,z)) \\ &= |\alpha\beta \sqcap \gamma|\end{aligned}$$

and

$$\begin{aligned}|\beta \sqcap \alpha^\sharp\gamma| &\leq \textstyle\sum_{x\in X, y\in Y, z\in Z}(\alpha^\sharp(y,x) \wedge \gamma(x,z) \wedge \beta(y,z)) \\ &= \textstyle\sum_{x\in X, y\in Y, z\in Z}(\alpha(x,y) \wedge \beta(y,z) \wedge \gamma(x,z)) \\ &= |\alpha\beta \sqcap \gamma|\end{aligned}$$

and we are done. ■

So all conclusions we showed for boolean relations with the help of the Dedekind inequality hold also for fuzzy relations, particularly fuzzy relations fulfill the corollaries 2.2 and 2.3. So for all point relations x holds because of $\tau(x) = x^\sharp x$ the identity

$$|\tau(x)\alpha| = |x^\sharp x\alpha| = |x\alpha|$$

and for boolean relations $\rho : I \rightharpoondown X$ and fuzzy relations $\alpha : X \rightharpoondown Y$ holds

$$|\tau(\rho)\alpha| = |\alpha^\sharp \sqcap \rho^\sharp \nabla_{IY}|$$

Similarly to before we can characterise the cardinality of fuzzy relations by the properties of a fymliy of mappings:

Theorem 5.3: *A family of mappings $|\cdot| : Rel(X,Y) \to \mathbb{N}$ coincides with the cardinality of fuzzy relations iff it satisfies the following properties:*

(a) $|\alpha| = 0 \Leftrightarrow \alpha = 0_{XY}$

(b) $|id_I| = 1$ *and* $|\alpha^\sharp| = |\alpha|$

(c) $|\alpha \sqcup \alpha'| = |\alpha| + |\alpha'| - |\alpha \sqcap \alpha'|$*, particularly* $\alpha \sqsubseteq \alpha'$*, implies* $|\alpha| \leq |\alpha'|$

(d) (Dedekind inequality) If α *is univalent, then* $|\beta \sqcap \alpha^\sharp\gamma| \leq |\alpha\beta \sqcap \gamma|$ *and* $|\alpha \sqcap \gamma\beta^\sharp| \leq |\alpha\beta \sqcap \gamma|$ *hold.*

(e) $|k \cdot \alpha| = k \cdot |\alpha|$ *for all* $k \in [0,1]$

Proof: It is clear, that the cardinality of relations satisfies the properties above. To show the reverse direction we investigate a family of mappings $|\cdot|$ fulfilling the five requirements above. Then we can conclude analogously to the boolean case and obtain the equality

$$|\alpha| = \textstyle\sum_{x\in X} \sum_{y\in Y} |x\alpha y^\sharp|$$

The rest of the proof is simple calculating:

$$\begin{aligned}|\alpha| &= \textstyle\sum_{x\in X}\sum_{y\in Y}|x\alpha y^\sharp| \\ &= \textstyle\sum_{x\in X}\sum_{y\in Y}|\alpha(x,y)\cdot id_I| \\ &= \textstyle\sum_{x\in X}\sum_{y\in Y}\alpha(x,y) && \{\text{ properties (b) and (e) }\}\ \blacksquare\end{aligned}$$

For a fuzzy endorelation $\alpha : X \rightharpoondown X$ and a natural number $n \in \mathbb{N}$ we define a fuzzy endorelation α^n inductively by $\alpha^0 = id_X$ and $\alpha^{n+1} = \alpha^n\alpha$. If we interpret α as a capacity constraint on a system of pipes then $\alpha^n(x,y)$ is the maximum amount we can send on a path of length exactly n from x to y.

The reflexive and transitive hull $\alpha^* : X \rightharpoondown X$ is defined by $\alpha^* = \bigsqcup_{n\geq 0}\alpha^n$. In the interpretation above $\alpha^*(x,y)$ corresponds with the maximum amount we can send on a path of arbitrary length from x to y. An elementary argument shows by counting, that $\alpha^* = \bigsqcup_{0\leq n\leq |X|-1}\alpha^n$.

Now we move already in the direction of network flows. First we show the following

Lemma 5.4: *Let $\alpha : X \rightharpoondown X$ be a fuzzy relation and let s and t be two distinct elements of X. Then exists a fuzzy relation $\xi : X \rightharpoondown X$ with the following properties:*

(a) $\xi \sqsubseteq \alpha$, $s\xi^\sharp = t\xi = 0_{IX}$ *and* $|s\xi| = |s\alpha^* t^\sharp|$

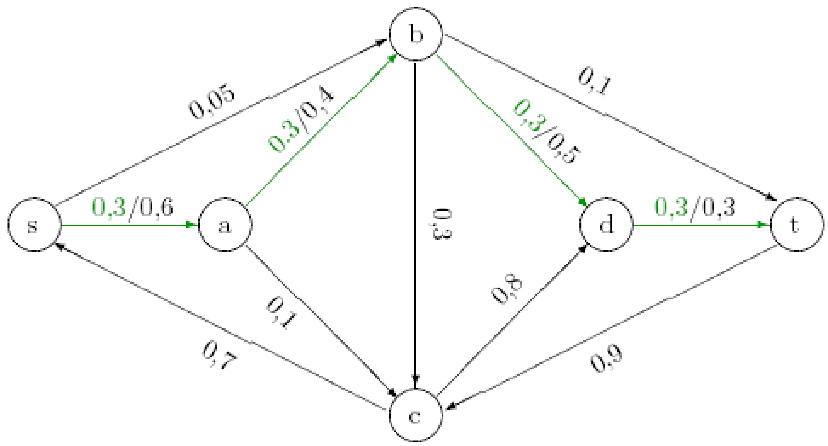
(b) $|\xi \sqcap \rho_0^\sharp\nabla_{IX}| = |\xi^\sharp \sqcap \rho_0^\sharp\nabla_{IX}|$ *for all boolean relations* $\rho_0 : I \rightharpoondown X$*. mit* $(s \sqcup t) \sqcap \rho_0 = 0_{IX}$*. Equivalent is the formulation* $|\tau\xi| = |\tau\xi^\sharp|$ *for all tests on* X *with* $\tau \sqsubseteq \tau((s \sqcup t)^-)$

(c) $\xi \sqcap \xi^\sharp = 0_{XX}$

Before we begin to prove we will try to interpret the properties of ξ. As usual we interpret the values of α as capacity constraint on the edges of a graph with X as its set of nodes. s plays the role of a source, from where we send something out, and t becomes a sink t, where this is sent to.

Part (a) states, that ξ respects the capacity constraint ($\xi \sqsubseteq \alpha$), that nothing is sent back to s and nothing is sent out of t ($s\xi^\sharp = t\xi = 0_{IX}$) and that ξ sends an amount modelled by $|s\alpha^* t|$. Part (b) states, that for every subset of X, which contains neither s nor t, summarised over all nodes the same quantum enters such a subset as it leaves. With other words, only at s can be something created, and only at t something can disappear. (c) prevents the flow-back to a node already sending out.

In the following depiction the already introduced fuzzy relation is coloured black and described with black figures on the edges. A fuzzy relation with the properties from the previous lemma overlays the edges along its way in green and has at the position, where its value greater than zero, green figures.



Proof: Let $k = |s\alpha^* t^\sharp| = \alpha^*(s,t)$. Because of $\alpha^* = \bigsqcup_{0\leq n\leq |X|-1} \alpha^n$ exists a $p < n$ and a sequence $s = v_0, v_1, \cdots, v_p = t$ of elements in X with $k = \bigwedge_{j=1}^{p} \alpha(v_{j-1}, v_j)$. A fuzzy relation $\xi : X \rightharpoonup X$ satisfying the properties of the lemma can be obtained by $\xi = k \cdot (\bigsqcup_{j=1}^{p} v_{j-1}^\sharp v_j)$. Then (c) is clear because of the distinctness of the v_j's, and with the same reason $s\xi^\sharp = t\xi^\sharp = 0_{IX}$ holds. $|s\xi| = |s\alpha^* t^\sharp|$ id fulfilled by construction and by calculating we show

$$
\begin{aligned}
k \cdot (v_{j-1}^\sharp v_j) &\sqsubseteq \alpha(v_{j-1}, v_j) \cdot (v_{j-1}^\sharp id_I v_j) \\
&= v_{j-1}^\sharp (\alpha(v_{j-1}, v_j) \cdot id_I) v_j \\
&= v_{j-1}^\sharp v_{j-1} \alpha v_j^\sharp v_j \\
&\sqsubseteq \alpha
\end{aligned}
$$

and hence $\xi \sqsubseteq \alpha$.

For the proof of part (b) we note first, that for all point relations $x : I \rightharpoonup X$ the equality $|\tau(x)\xi| = |\tau(x)\xi^\sharp|$ holds: for $x \neq v_1, v_2, \vdots, v_{p-1}$ we obtain zero on both sides, and in the case of $x \in \{v_1, v_2, \cdots, v_{p-1}\}$ according to the construction of ξ both sides are k. Another important fact is, that because of the distinctness of the v_i's both ξ and ξ^{sharp} are injective.

We show the claim by induction over the structure of τ. If $\tau = 0_{XX}$ or τ has the form $\tau(x)$ for a point relation x the claim is clear. Let now $\tau = \hat{\tau} \sqcup \tau(x)$, where x is a point relation with $\hat{\tau} \sqcap \tau(x) = 0_{XX}$ and the claim for $\hat{\tau}$ is as induction assumption already shown. Then holds:

$$
\begin{aligned}
&|\tau\xi| = \\
&\qquad \{ \text{ definition of } \tau \ \} \\
&|(\hat{\tau} \sqcup \tau(x))\xi| = \\
&\qquad \{ \ (\alpha \sqcup \beta)\gamma = \alpha\gamma \sqcup \beta\gamma \ \} \\
&|\hat{\tau}\xi \sqcup \tau(x)\xi| = \\
&\qquad \{ \ |\alpha \sqcup \beta| = |\alpha| + |\beta| - |\alpha \sqcap \beta|
\end{aligned}
$$

$|\hat{\tau}\xi| + |\tau(x)\xi| - |\hat{\tau}\xi \sqcap \tau(x)\xi| =$
$\{\ \xi \text{ injective} \Rightarrow (\alpha \sqcap \beta)\xi = \alpha\xi \sqcap \beta\xi\ \}$
$|\hat{\tau}\xi| + |\tau(x)\xi| - |(\hat{\tau} \sqcap \tau(x))\xi| =$
$\{\ \hat{\tau} \sqcap \tau(x) = 0_{XX}\ \}$
$|\hat{\tau}\xi| + |\tau(x)\xi| =$
$\{\ \text{induction assumption } |\tau(x)\xi| = |\tau(x)\xi^\sharp|\ \}$
$|\hat{\tau}\xi^\sharp| + |\tau(x)\xi^\sharp| =$
$\{\ \{\ \hat{\tau} \sqcap \tau(x) = 0_{XX}\ \}$
$|\hat{\tau}\xi^\sharp| + |\tau(x)\xi^\sharp| - |(\hat{\tau} \sqcap \tau(x))\xi^\sharp| =$
$\{\ \xi^\sharp \text{ injective} \Rightarrow (\alpha \sqcap \beta)\xi^\sharp = \alpha\xi^\sharp \sqcap \beta\xi^\sharp\ \}$
$|\hat{\tau}\xi^\sharp| + |\tau(x)\xi^\sharp| - |\hat{\tau}\xi^\sharp \sqcap \tau(x)\xi^\sharp| =$
$\{\ |\alpha \sqcup \beta| = |\alpha| + |\beta| - |\alpha \sqcap \beta|$
$|\hat{\tau}\xi^\sharp \sqcup \tau(x)\xi^\sharp| =$
$\{\ (\alpha \sqcup \beta)\gamma = \alpha\gamma \sqcup \beta\gamma\ \}$
$|(\hat{\tau} \sqcup \tau(x))\xi^\sharp| =$
$\{\ \text{definition of } \tau\ \}$
$|\tau\xi^\sharp|$ ■

A relation like the one constructed in the proof with the properties

(i) $\xi \sqsubseteq \alpha$

(ii) for each node $x \in X$ exists at most one edge (x, y) with $\xi(x, y) > 0$ and at most one edge (y, x) with $\xi(y, x) > 0$

(iii) for all tests τ on X with $\tau \sqsubseteq \tau((s \sqcup t)^-)$ holds $|\tau\xi| = |\tau\xi^\sharp|$

is called a *path flow* on α from s to t.

6 Network Flows

6.1 Networks and Pseudonetworks

A network in old fashioned sense is a graph (V, E, s, t, c) with a set of nodes V and a set of edges E, where s (the source) and t (the sink) are two distinct nodes and $c : E \to \mathbb{R}_0^+$ is the so-called capacity function. However, we can w.l.o.g. the range of c reduce to [0,1] by dividing all capacities with $C := max\{c(e)|e \in E\}$. So the following definition of networks in the sense of fuzzy relation is motivated:

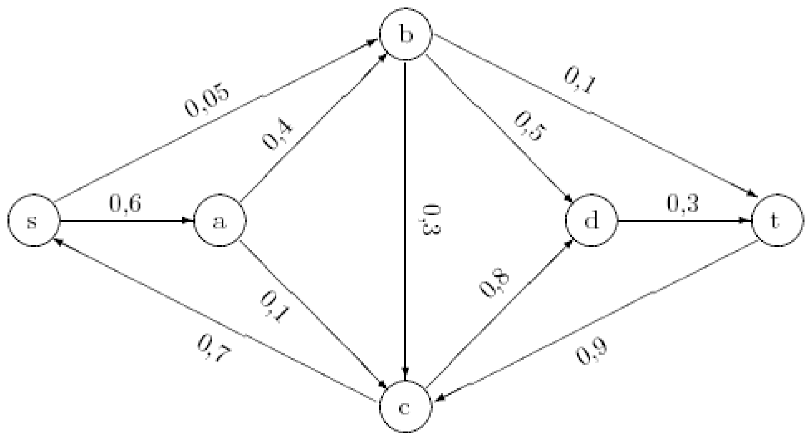
Definition (Network): A *network* N is a triple $N = (\alpha : X \rightharpoondown X, s, t)$ consisting of a fuzzy relation $\alpha : X \rightharpoondown X$ and two distinct elements s (the *source*) and t (the *sink*) of X, and where α satisfies the property

$$\alpha \sqcap \alpha^\sharp = 0_{XX}$$

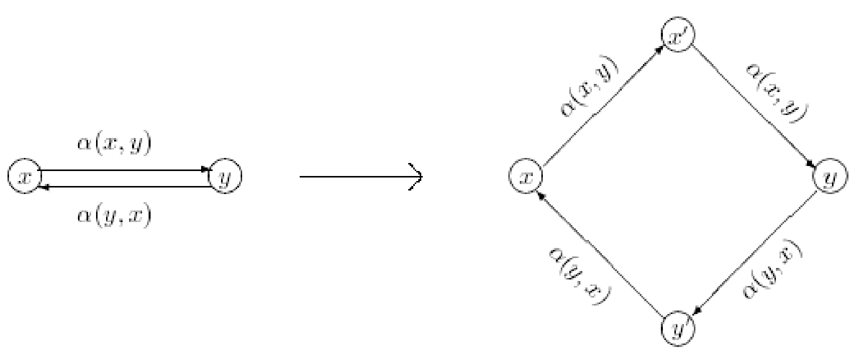
For a pair $(x, y) \in X \times Y$ we call $\alpha(x, y)$ the *capacity* of (x, y).

Contrary to [Kaw] we give up the requirements $s\alpha^\sharp{=}0_{XI}$ (correct would be $s\alpha^\sharp = 0_{IX}$) and $t\alpha = 0_{IX}$. For this win of generality we will get problems, which will be solved by the introduction of the net flow.

A graphic representation of a network is shown in the following depiction; we know it already from before:



If the requirement $\alpha \sqcap \alpha^\sharp = 0_{XX}$ is not satisfied we call N a *pseudonetwork*. This requirement may look as too strong, but if we are confronted with a fuzzy relation $\alpha : X \rightharpoondown X$ not fulfilling this condition we can construct a new network $\hat{N} = (\hat{\alpha} : \hat{X} \rightharpoondown \hat{X}, s, t)$ as follows: for all pairs $(x, y) \in X \times Y$ with $\alpha(x, y) \sqcap \alpha^\sharp(x, y) \neq 0$ we introduce two additional elements x' and y' and set $\hat{\alpha}(x, y) = \hat{\alpha}(y, x) = \hat{\alpha}(x', y') = \hat{\alpha}(y', x') = 0$, $\hat{\alpha}(x, x') = \hat{\alpha}(x', y) = \alpha(x, y)$, $\hat{\alpha}(y, y') = \hat{\alpha}(y', x) = \alpha(y, x)$ and $\hat{\alpha}(x', x) = \hat{\alpha}(y, x') = \hat{\alpha}(y', y) = \hat{\alpha}(x, y') = 0$. Because of the later introduced flow conservation it is clear, that a one-to-one correspondence between flows in the original network and the modified network exists. This method is shown in the next depiction.



6.2 Flows in Networks

Definition (Flow): A *flow* φ in a pseudonetwork $N = (\alpha : X \rightharpoondown X, s, t)$ is a fuzzy relation $\varphi : X \rightharpoondown X$, so that $\varphi \sqsubseteq \alpha$ and $|\tau_0\varphi| = |\tau_0\varphi^\sharp|$ for all test relations $\tau_0 : X \rightharpoondown X$ with $\tau_0 \sqsubseteq (\tau(X\backslash\{s,t\}))$ holds. Because of the equality $|\tau_0\varphi^\sharp| = |(\tau_0\varphi^\sharp)^\sharp| = |\varphi\tau_0|$ (note, that τ_0 as a test has the property $\tau_0 = \tau_0^\sharp$) this is equivalent to the more intuitive requirement $|\tau_0\varphi| = |\varphi\tau_0|$ ("what reaches all together all nodes in τ_0 has to leave them also").

Because of the identity $\tau(\rho)\ \nabla_{XY} = \rho^\sharp\nabla_{IY}$ for tests $\tau : X \rightharpoondown X$ and boolean relations $\rho : I \rightharpoondown X$ this definition is equivalent to the requirement $|\varphi \sqcap \rho_0^\sharp\nabla_{IX}| = |\varphi^\sharp \sqcap \rho_0^\sharp\nabla_{IX}|$ for all boolean relations $\rho_0 : I \rightharpoondown X$ with $\rho_0 \sqsubseteq (s \sqcup t)^-$. We will choose this version, if it is advantageous in the current algebraic context.

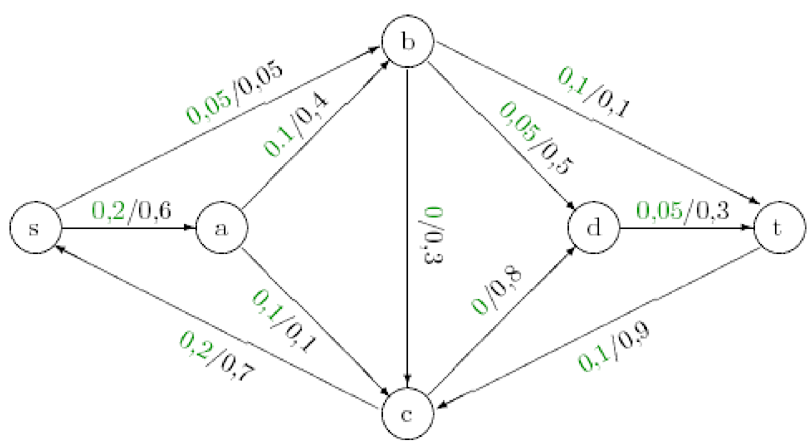
The first part of this definition is the so-calles *capacity constraint*: over an edge the flow can be at most as high as allowed by the capacity of the edge. The second part corresponds to the *flow preservation*, commonly written as

$$\sum_{v\in V} \varphi(v,u) = \sum_{v\in V} \varphi(u,v)\ \forall u \in X \setminus \{s,t\}$$

(see e.g. [Jun, p.147]) Our version seems to be sharper than the common one: it requires, that for every subset of $X\backslash\{s,t\}$ the sum of the outflows ($|\tau_0\varphi|$) equals the sum of the inflows ($|\varphi\tau_0|$). But these two definitions are equivalent: it is clear, that our definition implies the common one; one has to choose for τ_0 only a test $\tau(x)$ belonging to a point relation x. That the usual definition implies our definition can be seen similarly to the proof of lemma 3.2(b).

A comparison of the two formulations $|\tau_0\varphi^{|}|=|\varphi\tau_0|$ and the original $|\varphi\sqcap\rho_0^\sharp\nabla_{IX}|=|\varphi^\sharp\sqcap\rho_0^\sharp\nabla_{IX}|$ from [Kaw, p.10] shows the advantage of test: the new version is much more intuitive and also algebraically more comfortable.

A flow on our example network is shown in the next depiction.



The capacities are marked with black figures, the value of the flow with green ones.

For a flow φ in a network holds because of $\varphi \sqsubseteq \alpha$ and $\alpha \sqcap \alpha^\sharp = 0_{XX}$ the property $\varphi \sqcap \varphi^\sharp = 0_{XX}$.

Next we show some fundamental properties of network flows:

Proposition 6.1: *Let $N = (\alpha : X \rightharpoondown X, s, t)$ be a pseudonetwork. Then for every flow φ holds the equality $|s\varphi| - |s\varphi^\sharp| = |t\varphi^\sharp| - |t\varphi|$*

Proof: Let $\tau_0 = \tau(X \backslash \{s,t\})$. Then holds:

$$
\begin{aligned}
&|\varphi| = \\
&\qquad \{\ id_X\alpha = \alpha\ \} \\
&|id_X\varphi| = \\
&\qquad \{\ \text{construction of } \tau_0\ \} \\
&|(\tau(s) \sqcup \tau(t) \sqcup \tau_0)\varphi| = \\
&\qquad \{\ \tau(s), \tau(t), \tau_0 \text{ disjoint}\ \} \\
&|\tau(s)\varphi| + |\tau(t)\varphi| + |\tau_0\varphi| = \\
&\qquad \{\ |\tau(x)\alpha| = |x\alpha| \text{ for pointrelation } x\ \} \\
&|s\varphi| + |t\varphi| + |\tau_0\varphi|
\end{aligned}
$$

and

$$
\begin{aligned}
&|\varphi^\sharp| = \\
&\qquad \{\ id_X\alpha = \alpha\ \} \\
&|id_X\varphi^\sharp| = \\
&\qquad \{\ \text{construction of } \tau_0\ \} \\
&|(\tau(s) \sqcup \tau(t) \sqcup \tau_0)\varphi^\sharp| = \\
&\qquad \{\ \tau(s), \tau(t), \tau_0 \text{ disjoint}\ \} \\
&|\tau(s)\varphi^\sharp| + |\tau(t)\varphi^\sharp| + |\tau_0\varphi^\sharp| = \\
&\qquad \{\ |\tau(x)\alpha| = |x\alpha| \text{ for point relation } x\ \} \\
&|s\varphi^\sharp| + |t\varphi^\sharp| + |\tau_0\varphi^\sharp|
\end{aligned}
$$

Therefore holds

$$|s\varphi| + |t\varphi| + |\tau_0\varphi| = |s\varphi^\sharp| + |t\varphi^\sharp| + |\tau_0\varphi^\sharp|$$

and hence because of the flow conservation $|\tau\varphi| = |\tau\varphi^\sharp|$

$$|s\varphi| + |t\varphi| = |s\varphi^\sharp| + |t\varphi^\sharp|$$

from which the claim follows. ■

For further investigations we need some definitions.

6.3 Cuts and Residual Networks

Definition (Value, Cuts and Residual Networks): Let $N = (\alpha : X \rightharpoondown X, s, t)$ be a pseudonetwork.

(a) The *value* $val(\varphi)$ of a flow φ on N is defined by $val(\varphi) = |s\varphi| - |s\varphi^\sharp| = |t\varphi^\sharp| - |t\varphi|$.

(b) A flow φ on N is called *maximal*, if $val(\varphi) \geq val(\psi)$ for all flows ψ on N.

(c) A *cut* ρ of N is a boolean relation $\rho : I \rightharpoondown X$ with $s \sqsubseteq \rho \sqsubseteq t^-$. A *test cut* τ of N is a test τ on X with $\tau(s) \sqsubseteq \tau \sqsubseteq \tau(s^-)$.

(d) The *capacity* $c(\rho)$ of a cut ρ is defined as $c(\rho) = |\alpha \sqcap \rho^\sharp \rho^-|$. Analogously Capacity $c(\tau)$ of a test cut τ is defined by $c(\tau) = |\tau\alpha\tau^c|$

(e) A cut (test cut) π of N is called *minimal*, if $c(\pi) \leq c(\sigma)$ holds for all cuts (test cuts) σ of N.

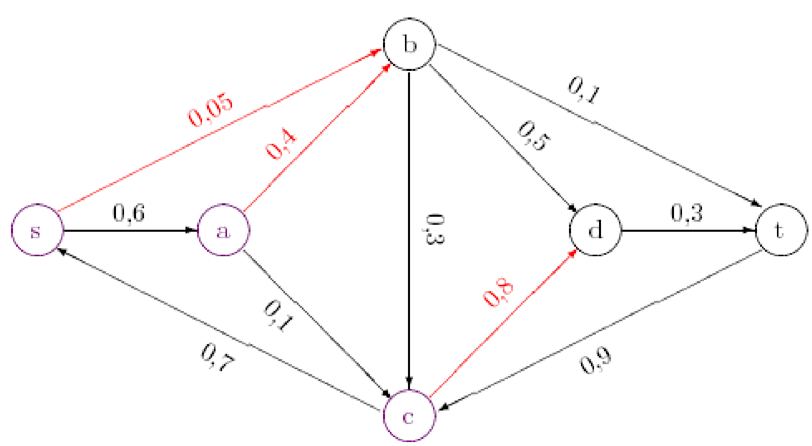
(f) For a flow φ on N we define a fuzzy relation $\varphi_\alpha : X \rightharpoondown X$ by $\varphi_\alpha = (\alpha \ominus \varphi) \sqcup \varphi^\sharp$.

To this definitions some remarks may be interesting:

(a) The value of a flow is in a natural way defined as the net outflow out of the source, what is according to the previous lemma the same as the net inflow into the sink. The value of the flow of the previous depiction is 0,05+0,2-0,2 = 0,05.

(b) This seemingly simple definition could contain an unpleasant surprise: we don't know, whether a maximal flow exists at all. If the set of all values of flows in a network is a open subset of the real numbers it is impossible to determine a maximal flow.

(c+d) A cut resp. a test cut corresponds to a subset of X containing s but not t. In the case of a cut this subset is represented by a boolean relation $\rho : I \rightharpoondown X$, the same subset is modelled as a test cut by $\tau(\rho)$. The capacity of a cut or a test cut is intuitively the amount of flow which can be sent over the "borders" of the cut resp. test cut on the nodes outside it. What we can imagine by this is shown in the next depiction.



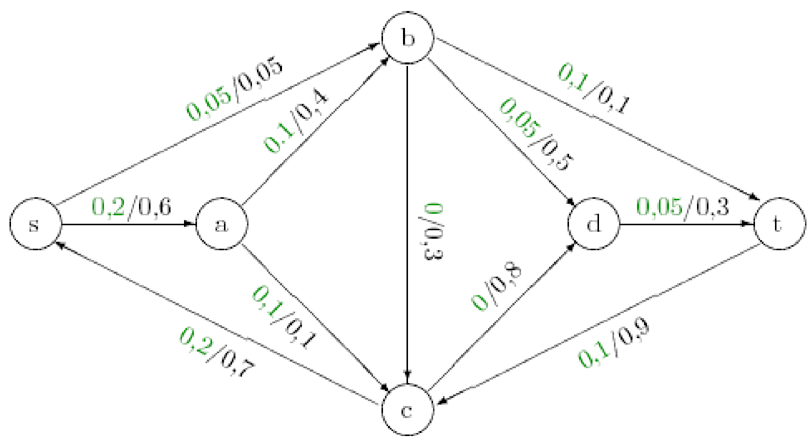
The cut $\rho = \{s, a, c\}$ is marked purple; the edges belonging to $\alpha \sqcap \rho^\sharp \rho^-$ are painted red. The capacity of this cut is 0,05+0,4+0,8 = 1,25.

An important observation is, that for a cut ρ the test $\tau(\rho)$ is a test cut, and that $c(\rho) = c(\tau(\rho))$ holds. Properties of cuts can therefore often be stated in a dual form as properties of test cuts and vice versa.

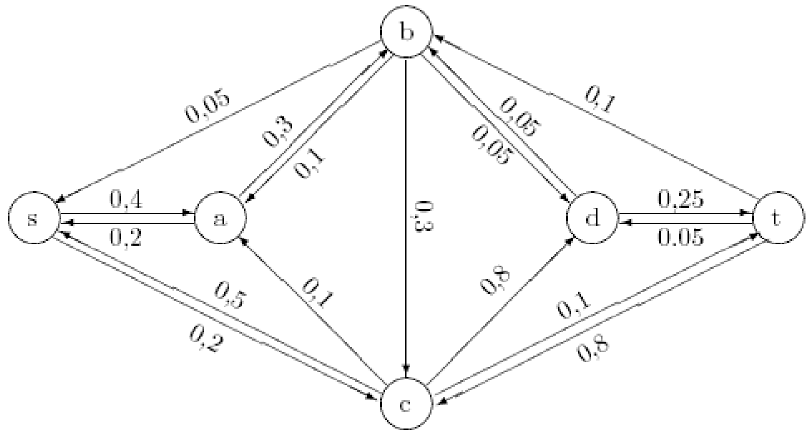
(e) Contrary to part (e) a minimum cut always exists, because there is only a finite number of them in a network.

(f) $\varphi_\alpha : X \rightharpoondown X$ defines the *residual network*. If over an edge (x, y) with capacity $\alpha(x, y)$ a flow of amount $\varphi(x, y)$ is sent, then over $(x, y,)$ can be sent a additionally amount of at most $\alpha(x, y) - \varphi(x, y)$ (note that $\alpha \ominus \varphi = \alpha - \varphi$ holds because of $\varphi \sqsubseteq \alpha$!). On the other hand it is possible to reduce the down to zero, what corresponds with a increasing of $\varphi(x, y)$ over the opposite edge (y, x) and is responsible for the term $\varphi^\sharp$. $\varphi_\alpha : X \rightharpoondown X, s, t$ is in general no network, but after all still a pseudonetwork.

For the sake of intuition we give once again the a flow on our example network



and the residual network induced by it:



From s to a we have an edge with capacity 0,4, because we used up already a share of 0,2 of the initial capacity 0,6. New is the edge from a to s; it has capacity 0,2, because we can reduce the flow from s to a by 0,2; in this case we would increase the flow over the opposite edge by 0,2.

The next proposition shows us the connections between cuts, residual networks and values of flows:

Proposition 6.2: *Let $N = (\alpha : X \rightharpoondown X, s, t)$ be a network. For all flows φ on N and all test cuts τ of N holds the equality*

$$val(\varphi) = |\tau\alpha\tau^c| - |\tau\varphi_\alpha\tau^c|$$

Analogously holds for for all cuts ρ the equality

$$val(\varphi) = |\alpha \sqcap \rho^\sharp\rho^-| - |\varphi_\alpha \sqcap \rho^\sharp\rho^-|$$

Proof: We show only the first claim; the second because of $\alpha\sqcap\rho^\sharp\rho^-=\tau(\rho)\alpha\tau(\rho)^c$.

First we observe, that $\varphi \sqsubseteq \alpha$ and therefore $(\alpha \ominus \varphi) \sqcap \varphi^\sharp \sqsubseteq \alpha \sqcap \alpha^\sharp = 0_{XX}$ hold. So we get

$$\tau\varphi_\alpha\tau^c = (\tau(\alpha \ominus \varphi)\tau^c) \sqcup \tau\varphi^\sharp\tau^c)$$

and hence because of proposition 4.1(d)

$$\tau\varphi_\alpha\tau^c = (\tau\alpha\tau^c \ominus \tau\varphi\tau^c) \sqcup \tau\varphi^\sharp\tau^c$$

and from proposition 5.1(b) follows, because additionally $(\alpha \ominus \varphi) \sqcap \varphi^\sharp \sqsubseteq 0_{XX}$ holds,

$$|\tau\varphi_\alpha\tau^c| = |\tau\alpha\tau^c| - |\tau\varphi\tau^c| + |\tau\varphi^\sharp\tau^c|$$

Let now be $\tau_0 = \tau(s^-) \sqcap \tau$. Then we can conclude as follows:

$val(\varphi) = |s\varphi| - |s\varphi^\sharp|$

$\{\ |s\alpha| = |\tau(s)\alpha)|\ \}$

$= |\tau(s)\varphi| - |\tau(s)\varphi^\sharp|$

$\{\ |\tau_0\varphi| = |\tau_0\varphi^\sharp|\ \}$

$= |\tau_0\varphi| + |\tau(s)\varphi| - |\tau_0\varphi^\sharp| - |\tau(s)\varphi^\sharp|$

{ disjointness of τ_0 and $\tau(s)$ }

$= |(\tau_0 \sqcup \tau(s))\varphi| - |(\tau_0 \sqcup \tau(s))\varphi^\sharp|$

{ construction of τ_0 }

$= |\tau\varphi| - |\tau\varphi^\sharp|$

$\{\ \alpha = \alpha id_X\ \}$

$= |\tau\varphi id_X| - |\tau\varphi^\sharp id_X|$

$\{\ \tau \sqcup \tau^c = id_X\ \}$

$= |\tau\varphi(\tau \sqcup \tau^c)| - |\tau\varphi^\sharp(\tau \sqcup \tau^c)|$

{ disjointness of τ and τ^c }

$= |\tau\varphi\tau| + |\tau\varphi\tau^c| - |\tau\varphi^\sharp\tau| - |\tau\varphi^\sharp\tau^c|$

$\{\ |\tau\alpha\tau| = |\tau\alpha^\sharp\tau|\ \}$

$= |\tau\varphi\tau^c| - |\tau\varphi^\sharp\tau^c|$

{ clear }

$= |\tau\alpha\tau^c| - (|\tau\alpha\tau^c| - |\tau\varphi\tau^c| + |\tau\varphi^\sharp\tau^c|)$

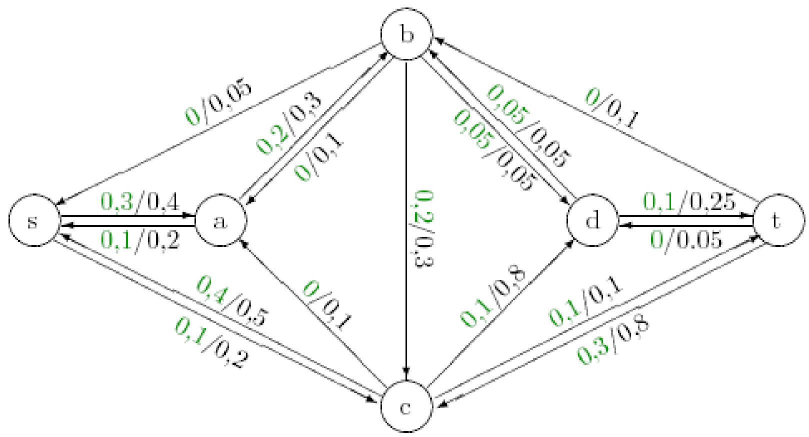
{ definition of φ_α }

$= |\tau\alpha\tau^c| - |\tau\varphi_\alpha\tau^c|$ ■

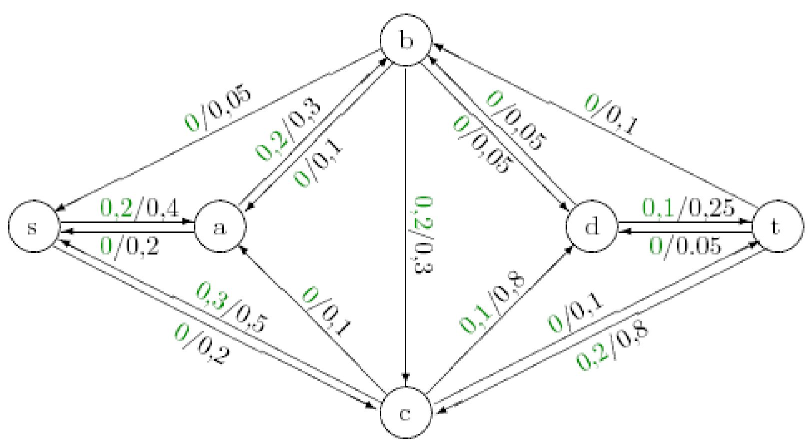
Definition (effective flow): Let $N = (\alpha : X \rightharpoonup X, s, t)$ be a pseudonetwork and φ a flow on N. Then the *effective flow* associated with φ is defined by $\varphi_e = \varphi \ominus \varphi^\sharp$.

The effective flow determines the amortised flow between two nodes. For a low in a network obviously $\varphi = \varphi_e$ holds.

Let's take now a look at a flow on our example network:



and the associated effective flow:



To justify the name effective flow we will convince ourselves, that φ_e is indeed a flow on N. Because of $\varphi \sqsubseteq \alpha$ holds also $\varphi \ominus \varphi^\sharp \sqsubseteq \alpha$, so the capacity constraint is satisfied. The flow conservation is shown only pointwise, i.e., we show only $|x\varphi_e| = |x\varphi_e^\sharp|$ for all point relations x unequal s or t. The flow conservation for arbitrary tests follows by induction over their structure, confirm the remarks after the definition of flow conservation.

We start with the flow conservation of φ for an arbitrary point relation x unequal s or t and calculate:

$$|x\varphi| = |x\varphi^\sharp| \Rightarrow$$

$$
\begin{array}{l}
\qquad\qquad\{ \text{ definition of } |\cdot| \ \} \\
\sum_{y\in X}\varphi(x,y) = \sum_{y\in X}\varphi^\sharp(x,y) \Rightarrow \\
\qquad\qquad\{ \text{ splitting of } X \text{ in disjoint subsets } \} \\
\sum_{y\in X,\varphi(x,y)>\varphi(y,x)}\varphi(x,y) + \sum_{y\in X,\varphi(x,y)\le\varphi(y,x)}\varphi(x,y) = \\
\sum_{y\in X,\varphi(x,y)>\varphi(y,x)}\varphi^\sharp(x,y) + \sum_{y\in X,\varphi(x,y)\le\varphi(y,x)}\varphi^\sharp(x,y) \Rightarrow \\
\qquad\qquad\{ \text{ elementary algebra } \} \\
\sum_{y\in X,\varphi(x,y)>\varphi(y,x)}\varphi(x,y) - \sum_{y\in X,\varphi(x,y)>\varphi(y,x)}\varphi^\sharp(x,y) = \\
\sum_{y\in X,\varphi(x,y)\le\varphi(y,x)}\varphi^\sharp(x,y) - \sum_{y\in X,\varphi(x,y)\le\varphi(y,x)}\varphi(x,y) \Rightarrow \\
\qquad\qquad\{ \text{ putting together of sums } \} \\
\sum_{y\in X,\varphi(x,y)>\varphi(y,x)}\varphi(x,y) - \varphi^\sharp(x,y) = \\
\sum_{y\in X,\varphi(y,x)\le\varphi(x,y)}\varphi^\sharp(x,y) - \varphi(x,y) \Rightarrow \\
\qquad\qquad\{ \text{ definition of } |\cdot| \text{ and } \ominus \} \\
|x(\varphi\ominus\varphi^\sharp)| = |x(\varphi^\sharp\ominus\varphi)| \Rightarrow \\
\qquad\qquad\{(\alpha\ominus\alpha^\sharp)^\sharp = (\alpha^\sharp\ominus\alpha)\} \\
|x(\varphi\ominus\varphi^\sharp)| = |x(\varphi\ominus\varphi^\sharp)^\sharp| \Rightarrow \\
\qquad\qquad\{ \text{ definition of } \varphi_e \} \\
|x\varphi_e| = |x\varphi_e^\sharp|
\end{array}
$$

The definition of the effective flow didn't take place only for itself, but it will be a useful tool. This can be seen at the next lemma:

Lemma 6.3: For a flow φ in a pseudonetnetwork holds the identity $val(\varphi) = val(\varphi_e)$.

Proof: First we show $|s\varphi y^\sharp| - |s\varphi^\sharp y^\sharp| = |s\varphi_e y^\sharp| - |s\varphi_e^\sharp y^\sharp|$ for all point relations $y : I \rightharpoondown X$:

$$
\begin{array}{l}
|s\varphi y^\sharp| - |s\varphi^\sharp y^\sharp| = \varphi(s,y) - \varphi^\sharp(s,y) \\
\qquad\{ \ (a\ominus b) - (b\ominus a) = a - b \ \} \\
\qquad\quad = (\varphi(s,y)\ominus\varphi^\sharp(s,y)) - (\varphi^\sharp(s,y)\ominus\varphi(s,y)) \\
\qquad\{ \text{ pointwise definition of } \ominus \ \} \\
\qquad\quad = (\varphi\ominus\varphi^\sharp)(s,y) - (\varphi^\sharp\ominus\varphi)(s,y) \\
\qquad\{ \text{ definition of } \varphi_e \ \} \\
\qquad\quad = \varphi_e(s,y) - \varphi_e^\sharp(s,y) \\
\qquad\{ \ \alpha(x,y) = |x\alpha y^\sharp| \\
\qquad\quad = |s\varphi_e y^\sharp| - |s\varphi_e^\sharp y^\sharp|
\end{array}
$$

Considering the equalities

$$val(\psi) = |s\psi| - |s\psi^\sharp| = |\psi \sqcap s^\sharp\nabla_{IX}| - |\psi^\sharp \sqcap s^\sharp\nabla_{IX}|$$

and

$$|\psi \sqcap s^\sharp\nabla_{IX}| = \sum_{x\sqsubseteq\nabla_{IX}} |\psi \sqcap s^\sharp x| = \sum_{x\sqsubseteq\nabla_{IX}} |s\psi x^\sharp|$$

we obtain the claim by summing the first shown equation over all $y \sqsubseteq \nabla_{IX}$. ■

6.4 Augmenting a Flow

In the next lemma we will investigate how we can manipulate flows and how the values of them are changed by our manipulation. Our aim is, to increase the values of a flow to get a flow of a maximum value.

Lemma 6.4: *Let $N = (\alpha : X \rightharpoondown X, s, t)$ be a network and φ a flow on N. If $\xi : X \rightharpoondown X$ is a fuzzy relation with the properties $\xi \sqsubseteq \varphi_\alpha$ and $|\xi \sqcap \rho_0^\sharp \nabla_{IX}| = |\xi \sqcap \rho_0^\sharp \nabla_{IX}|$ for all boolean relations $\rho_0 : I \rightharpoondown X$ with $\rho_0 \sqsubseteq (s \sqcup t)^-$ (with other words, ξ is a flow on the peudonetwork $(\varphi_\alpha : X \rightharpoondown X, s, t)$, the residual network of φ) then the fizzy relation*

$$\psi = (\varphi \ominus (\alpha \sqcap \xi^\sharp)) \oplus (\alpha \sqcap \xi)$$

is a flow on N with $val(\psi) = val(\varphi) + val(\xi)$.

In this we say, that we *augment* the flow φ by the flow ξ.

Proof: We show first the equality

$$(\varphi \oplus \xi) \ominus (\varphi \oplus \xi)^\sharp = (\varphi \ominus (\alpha \sqcap \xi^\sharp)) \oplus (\alpha \sqcap \xi)$$

For pairs $(x, y) \in X \times X$ mit $\alpha(x, y) > 0$ holds $\alpha^\sharp(x, y) = 0$, because N is a network and hence due to the capacity constraint also $\varphi^\sharp(x, y) = 0$. Therefore $\varphi^{alpha}(x, y)$ can be written as $\varphi_\alpha(x, y) = \alpha(x, y) - \varphi(x, y)$. So we obtain a chain of inequalities $\varphi(x, y) + \xi(x, y) \leq \varphi(x, y) + \varphi_\alpha(x, y) \leq 1$. According to the definition of a network holds $\alpha(y, x) = 0$ and hence $\varphi(y, x) = 0$ and $\varphi_\alpha(y, x) = \varphi(x, y)$, together we have $\varphi(y, x) + \xi(y, x) \leq \varphi(x, y) \leq \alpha(x, y) \leq 1$. For pairs (x, y) with $\alpha(x, y) = \alpha(y, x) = 0$ holds $\varphi(x, y) = \varphi(y, x) = \xi(x, y) = \xi(y, x) = 0$, therefore we can conclude, that $\varphi(x, y) + \xi(x, y) = (\varphi \oplus \xi)(x, y)$ holds for all $(x, y) \in X \times X$. To show the equality above we compare the two fuzzy relations pointwise, i.e., we show, that their values coincide on all pairs $(x, y) \in X \times X$. For pairs with $\alpha(x, y) = \alpha(y, x) = 0$ the claim is trivial; all involved fuzzy relations φ, α, ξ and φ_α have the value zero. Let therefore $(x, y) \in X$ and w.l.o.g. let $\alpha(x, y) > 0$. The holds:

$$\begin{aligned}
((\varphi \oplus \xi) \ominus (\varphi \oplus \xi)^\sharp)(x, y) &= (\varphi(x, y) + \xi(x, y)) \ominus (\varphi^\sharp(x, y) + \xi^\sharp(x, y)) \\
&\quad \{\ \alpha(x, y) \geq 0 \Rightarrow \alpha^\sharp(x, y) = 0 \Rightarrow \varphi^\sharp(x, y) = 0\ \} \\
&= (\varphi(x, y) + \xi(x, y)) \ominus \xi^\sharp(x, y) \\
&\quad \{\ \xi^\sharp(x, y) \leq \varphi_\alpha^\sharp(x, y) = \varphi(x, y)\ \} \\
&= \varphi(x, y) + \xi(x, y) - \xi^\sharp(x, y)
\end{aligned}$$

and because of $\xi(x, y) \leq \varphi_\alpha(x, y) \leq \alpha(x, y)$ and $\xi^\sharp(x, y) \leq \varphi_\alpha(x, y) \leq \varphi(x, y) \leq \alpha(x, y)$ we can conclude

$$\begin{aligned}
((\varphi \ominus (\alpha \sqcap \xi^\sharp)) \oplus (\alpha \sqcap \xi))(x, y) &= ((\varphi \ominus \xi^\sharp) \oplus \xi)(x, y) \\
&= (\varphi(x, y) \ominus \xi^\sharp(x, y)) \oplus \xi(x, y) \\
&= (\varphi(x, y) - \xi^\sharp(x, y)) \oplus \xi(x, y) \\
&= (\varphi(x, y) - \xi^\sharp(x, y)) + \xi(x, y)
\end{aligned}$$

The two fuzzy relations coincide therfore on (x, y). For the pair (y, x) holds $\alpha(y, x) = 0$ because of the definition of a network and hence $\varphi(y, x) = 0$. So we see, that

$$((\varphi \ominus (\alpha \sqcap \xi^\sharp)) \oplus (\alpha \sqcap \xi))(y, x) = 0$$

holds. For the left side we can argue as follows:

$$\begin{aligned} ((\varphi \oplus \xi) \ominus (\varphi \oplus \xi)^\sharp)(x, y) &= (\xi \ominus (\varphi^\sharp \oplus \xi^\sharp))(x, y) \\ &= \xi(x, y) \ominus (\varphi^\sharp(x, y) + \xi^\sharp(x, y)) \\ &= 0 \end{aligned}$$

So we showed the equality of the two fuzzy relations. The properties of the value of the flow follow from the previous lemma, we have only to show, that ψ is really a flow on N. First we observe, that

$$\begin{aligned} \xi^\sharp \sqcap \alpha &\sqsubseteq (\alpha^\sharp \sqcup \varphi) \sqcap \alpha \\ &= \varphi \end{aligned}$$

and

$$\begin{aligned} \xi \sqcap \alpha &\sqsubseteq ((\alpha \ominus \varphi) \sqcup \varphi^\sharp) \sqcap \alpha \\ &= \alpha \ominus \varphi \end{aligned}$$

hold. Because of $\varphi \sqsubseteq \alpha$ and $\xi \sqcap \alpha \sqsubseteq \alpha \ominus \varphi$ holds also

$$\psi \sqsubseteq \varphi \oplus (\xi \sqcap \alpha) \sqsubseteq \alpha$$

So we showed that ψ respects the capacity constraint, the last thing we have to do is to prove the flow conservation. Let $\rho_0 : I \rightharpoonup X$ be a boolean relation with $\rho_0 \sqsubseteq (s \sqcup t)^-$ and set $\hat{\rho}_0 = \rho_0^\sharp \nabla_{IX}$. As a boolean relation ρ_0 satisfies

$$\psi \sqcap \hat{\rho}_0 = ((\varphi \sqcap \hat{\rho}_0) \ominus (\alpha \sqcap \xi^\sharp \sqcap \hat{\rho}_0)) \oplus (\alpha \sqcap \xi \sqcap \hat{\rho}_0)$$

because of proposition 5.1(b) and from proposition 5.1(c) follows

$$|\psi \sqcap \hat{\rho}_0| = |\varphi \sqcap \hat{\rho}_0| - |(\alpha \sqcap \xi^\sharp \sqcap \hat{\rho}_0| + |\alpha \sqcap \xi \sqcap \hat{\rho}_0|$$

because of $\varphi \ominus (\alpha \sqcap \xi^\sharp) \sqsubseteq \alpha$, $\alpha \sqcap \xi \sqsubseteq \alpha \ominus (\varphi \ominus (\alpha \sqcap \xi^\sharp))$ and $\alpha \sqcap \xi^\sharp \sqsubseteq \varphi$. If we remember that $\xi \sqsubseteq \alpha \sqcup \alpha^\sharp$ we obtain

$$|\psi \sqcap \hat{\rho}_0| - |\xi \sqcap \hat{\rho}_0| = |\varphi \sqcap \hat{\rho}_0| - |\alpha \sqcap \xi^\sharp \sqcap \hat{\rho}_0| - |\alpha^\sharp \sqcap \xi \sqcap \hat{\rho}_0|$$

Analogous considerations lead to

$$|\psi^\sharp \sqcap \hat{\rho}_0| - |\xi^\sharp \sqcap \hat{\rho}_0| = |\varphi^\sharp \sqcap \hat{\rho}_0| - |\alpha^\sharp \sqcap \xi \sqcap \hat{\rho}_0| - |\alpha \sqcap \xi^\sharp \sqcap \hat{\rho}_0|$$

Because of the flow properties of φ an ξ both $|\varphi \sqcap \hat{\rho}_0| = |\varphi^\sharp \sqcap \hat{\rho}_0|$ and $|\xi \sqcap \hat{\rho}_0| = |\xi^\sharp \sqcap \hat{\rho}_0|$ hold, wherefrom obviously together with the previous two equations follows $|\psi \sqcap \hat{\rho}| = |\psi^\sharp \sqcap \hat{\rho}_0|$, what implies the flow conservation for ψ. ■

6.5 Max-Flow-Min-Cut-Theorem

Now we show the famous max-flow-min-cut-theorem, which characterises the maximality of flows, with our fuzzy relational tools.

Theorem 6.5 (Max-Flow-Min-Cut-Theorem): *Let $N = (\alpha : X \rightharpoondown X, s, t)$ be network and φ a flow on N. Then are equivalent:*

(a) φ is maximal

(b) $t \sqcap s\varphi_\alpha^* = 0_{IX}$, or equivalently $|s\varphi_\alpha^* t^\sharp| = 0$

(c) It exists a cut ρ with $val(\varphi) = |\alpha \sqcap \rho^\sharp \rho^-|$

Proof:

(a)⇒(b): Let φ be maximal and $k = |s\varphi_\alpha^* t^\sharp|$. According to Lemma 1 exists a fuzzy relation $\xi : X \rightharpoondown X$ with $\xi \sqsubseteq \varphi_\alpha$, $s\xi^\sharp = 0_{IX}$, $|s\xi| = k$ and $|\xi \sqcap \rho_0^\sharp \nabla_{IX}| = |\xi^\sharp \sqcap \rho_0^\sharp \nabla_{IX}|$ for all boolean relations $\rho : I \rightharpoondown X$ with $\rho \sqsubseteq (s \sqcup t)^-$. That means, ξ is a flow on the pseudo network φ_α with value $val(\xi) = |s\xi| = k$ (note, that $|s\xi^\sharp| = 0$!). According to the previous lemma

$$\psi = (\varphi \ominus (\alpha \sqcap \xi^\sharp)) \oplus (\alpha \sqcap \xi)$$

is a flow on N with value $val(\psi) = val(\varphi) + val(\xi) = val(\varphi) + k$. Because φ is maximal $k = 0$ holds and hence the claim follows.

Intuitively means this, that it is impossible to send flow from s to t in the residual network, because otherwise we would obtain a flow with a higher value.

(b)⇒(c): Let $|t \sqcap s\varphi_\alpha^*| = 0_{IX}$. Then $\rho = (s\varphi_\alpha^*)^\bullet$ is a boolean relation from I into X, and according to the definition of α^* holds $s \sqsubseteq \rho$ and from $t \sqcap s\varphi_\alpha^*| = 0_{IX}$ follows $\rho \sqsubseteq t^-$; this means ρ is a cut of N. Furthermore we consider the inequality

$$\rho\varphi_\alpha \sqsubseteq (s\varphi_\alpha^*)^\bullet \varphi_\alpha^\bullet = (s\varphi_\alpha^* \varphi_\alpha)^\bullet \sqsubseteq (s\varphi_\alpha^*)^\bullet = \rho$$

So it is clear, that $\varphi_\alpha \sqcap \rho^\sharp \rho^- = 0_{IX}$ holds; intuitively spoken one can leave ρ only via edges (x, y) with $\varphi_\alpha(x, y) = 0$. Because of proposition 6.2 holds then

$$val(\varphi) = |\alpha \sqcap \rho^\sharp \rho^-| - |\varphi_\alpha \sqcap \rho^\sharp \rho^-| = |\alpha \sqcap \rho^\sharp \rho^-|$$

(c)⇒(a): Let ρ be a cut of N with $val(\varphi) = val(\rho) = |\alpha \sqcap \rho^\sharp \rho^-|$ and let ψ be an arbitrary flow on N. Then holds:

$$\begin{aligned} val(\psi) &= |\alpha \sqcap \rho^\sharp \rho^-| - |\psi_\alpha \sqcap \rho^\sharp \rho^-| \\ &\leq |\alpha \sqcap \rho^\sharp \rho^-| \\ &= val(\varphi) \ \blacksquare \end{aligned}$$

Still we don't know anything about he existence of a maximal flow in network, but for a certain class of networks we can show the existence of a maximal flow by construction.

Definition: A fuzzy relation $\alpha : X \rightharpoondown X$ is called *M-valued*, if its range is contained in the set $B_M := \{0, \frac{1}{M-1}, \frac{2}{M-1}, \ldots, 1\}$. Obviously a fuzzy relation is 2-values iff it is boolean.

Theorem 6.6: *Let $N = (\alpha : X \rightharpoondown X, s, t)$ be a network with an M-valued capacity function α. Then N has also an M-valued maximal flow. Such a flow can be computed in O(|E|·log((M-1))·T^*(|X|,2·|E|)) time, where $|E|$ denotes the number of edges of N and $T^*(n,m)$ is the time in which a maximal $s-t$-pathflow in a network with n nodes an m edges can be computed.*

Proof: As proof we give an algorithm which produces a flow with the desired properties (maximal and M-valued). It constructs a sequence of flows φ_i as follows:

(I) Set $\varphi_0 = 0_{XX}$. Obviously φ_0 is a flow on N.

(II) After computing φ_i set $k_i = |s((\alpha \ominus \varphi_i) \sqcup \varphi_i^\sharp)^* t^\sharp| = |s{\varphi_i}_\alpha t^\sharp|$. If $k_i = 0$ then φ_i is maximal according to the max-flow-min-cut-theorem, the algorithm returns φ_i and terminates. If $k_i > 0$ then apply the construction from the proof of part (a)⇒(b) of theorem 6.5. This yields a flow φ_{i+1} with value $val(\varphi_{i+1}) = val(\varphi_i) + k_i$.

Because M-values fuzzy relations are obviously closed under all relevant operations k_i is either zero or greater-equal $\frac{1}{M-1}$. The Algorithm increases therefore in every step the value of the flow by at leat $\frac{1}{M-1}$ or terminates with a maximal flow. The augmentation always takes place along a $s-t$−path with maximal capacity. For the runtime bound see [AMO, p.219 f.].

7 Application on Graph Theory

A boolean relations α from a set X into a set Y with $X \cap Y = \emptyset$ can be set in relation with a undirected bipartite graph. Therefore we choose as set of nodes $X \cdot\cup Y$ and the graph contains a edge iff (x, y) is contained in the relation resp. $\alpha(x, y) = 1$ holds. Our relational algebraic tools are very suited for proving theorems about undirected bipartite graphs in an algebraic calculating manner.

A lot of theorems contain the cardinality of subsets of X or Y; in our relational encoding we identify subsets with boolean relations $\rho : I \rightharpoondown X$ resp. $\rho : I \rightharpoondown Y$ and describe the cardinality of such a set by $|\rho|$. The cardinality of X and Y is given then by $|\nabla_{IX}|$ and $|\nabla_{IY}|$.

The representation of subsets by boolean relations $\rho : I \rightharpoondown X$ resp. $\rho : I \rightharpoondown Y$ has one advantage compared with the representation by the associates test $\tau(\rho)$: often we want to reason about the number of elements reachable from ρ under α. This number is easily described by $|\rho\alpha|$. The term $|\tau(\rho)\alpha|$ yields a too big value: it counts the number of edges out of ρ, not the number of reached nodes.

Matchings in common graph theory are subgraphs of a graph in such a way, that to edges of this subgraph don't have two endpoints in common. We called a relation $f : X \rightharpoondown X$ a matching, if it is a partial bijection, i.e., if $ff^\sharp \sqsubseteq id_X$ and $f^\sharp f \sqsubseteq id_Y$ hold. This corresponds to the requirement, that two edges don't have two common points; a matching as subgraph will be soon defined in relational vocabulary.

7.1 Nameless Theorems

First we consider some simpler theorems about matchings. For a boolean relation $\alpha : X \rightharpoondown X$ we call a matching $f : X \rightharpoondown X$ a *matching of* α, if $f \sqsubseteq \alpha$ holds.

Theorem 7.1: *Let $X \cap Y = \emptyset$, $\alpha : X \rightharpoondown Y$ a boolean relation and $f : X \rightharpoondown X$ a matching of α. Then for all boolean relations $\rho : I \rightharpoondown X$ the inequality $|f| \leq |\nabla_{IX}| - (|\rho| - |\rho\alpha|)$ holds.*

Before the proof we will translate the theorem in common graph theoretic language.

The term ∇_{IX} models as a boolean relation the set X; $|\nabla_{IX}|$ therefore corresponds to the cardinality of X. By the relation ρ a subset of X is described; $\rho\alpha$ is a boolean relation from I into Y and corresponds to those elements of Y which are under α in relation with an element from the subset of X described by ρ. In graph theory this are all nodes reachable from ρ under α.

Expressed in terms of graph theory the theorem states:
Let $G = (X \dot{\cup} Y, E)$ be a undirect bipartite graph with set of nodes $X \dot{\cup} Y$ and edges E and let $E(X_1)$ be for a set of nodes $X_1 \subseteq X$ the set $\{ y \mid \exists x \in X_1 : \{x, y\} \in E \}$. Then holds for all matchings f in G and for all subsets $X_1 \subseteq X$ the inequality $|f| \leq |X| - (|X_1| - |E(X_1)|)$

Proof: For the prove we reactivate our matric representation of boolean relations. Let $\rho : I \rightharpoondown X$ be a boolean relation and $\tau : X \rightharpoondown X$ a test on X. A numeration of X is given in the form $\{x_1, x_2, ..., x_n\}$. ρ_i and τ_i are defined in a obvious way as $\rho_i = \rho(x_i)$ and $\tau_i = \tau(x_i)$. Then the matric associated with $\rho \sqcap \nabla_{IX}\tau$ can be calculated as follows:

$$\begin{pmatrix} \rho_1, & \rho_2, & \ldots, & \rho_n \end{pmatrix} \sqcap \begin{pmatrix} 1, & 1, & \ldots, & 1 \end{pmatrix} \cdot \begin{pmatrix} \tau_1 & \ldots & \ldots & 0 \\ \vdots & \tau_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \tau_n \end{pmatrix} =$$
$$\begin{pmatrix} \rho_1, & \rho_2, & \ldots, & \rho_n \end{pmatrix} \sqcap \begin{pmatrix} \tau_1, & \tau_2, & \ldots, & \tau_n \end{pmatrix} =$$
$$\begin{pmatrix} \rho_1 \sqcap \tau_1, & \rho_2 \sqcap \tau_2, & \ldots, & \rho_n \sqcap \tau_n \end{pmatrix}$$

So for all tests $\tau : X \rightharpoondown X$ and boolean relations $\rho : I \rightharpoondown X$ the equality $\rho \sqcap \nabla_{IX}\tau = \rho\tau$ and hence for all matchings $f : X \rightharpoondown X$ the relation $\rho \sqcap \nabla_{IX} f f^\sharp = \rho f f^\sharp$ hold(note that f as a matching satisfies $f f^\sharp \sqsubseteq id_X$ and that $f f^\sharp$ is a boolean relation).

After these preliminaries we obtain

$$\begin{aligned} |\rho \sqcap \nabla_{IX} f f^\sharp| &= |\rho f f^\sharp| && \{\rho \sqcap \nabla_{IX} f f^\sharp = \rho f f^\sharp\} \\ &= |\rho f| && \{\text{ corollary 2.2(d)}\} \end{aligned}$$

and

$$\begin{aligned} |\nabla_{IX} f f^\sharp| &= |\nabla_{IX} f| && \{\text{ corollary 2.2(d) }\} \\ &= |f| && \{\text{ corollary 2.3(a) }\} \end{aligned}$$

So the following chain of inequalities holds:

$$\begin{aligned} |\nabla_{IX}| &\geq |\rho \sqcup \nabla_{IX} f f^\sharp| && \{\rho \sqcap \nabla_{IX} f f^\sharp \sqsubseteq \nabla_{IX}\} \\ &= |\rho| + |\nabla_{IX} f f^\sharp| - |\rho \sqcap \nabla_{IX} f f^\sharp| && \\ &= |\rho| + |f| - |\rho f| && \{\text{s.o.}\} \\ &\geq |\rho| + |f| - |\rho\alpha| && \{f \sqsubseteq \alpha\} \end{aligned}$$

The claim follows then by simplest algebra. ■

Immediately follows now

Corollary 7.2: *Let $\alpha : X \rightharpoondown Y$ be a boolean relation and $X \cap Y = \emptyset$. Then for all matchings f of α holds the inequality*

$$|f| \leq |\nabla_{IX}| - \delta(\alpha)$$

where $\delta(\alpha) = max_{\rho : I \rightharpoondown X}(|\rho| - |\rho\alpha|)$

7.2 Hall's Theorem

Theorem 7.3 (Hall's Theorem): *Let $\alpha : X \rightharpoondown X$ be a boolean relation and X a nonempty set with $X \cap Y = \emptyset$. A total matching of α (i.e. an injective function $f : X \rightharpoondown Y$ with $f \sqsubseteq \alpha$) exists iff for all boolean relations $\rho : I \rightharpoondown X$ the inequality $|\rho| \leq |\rho\alpha|$ holds.*

Before the proof we translate the theorem in usual graph theoretic expressions (confirm the remarks to theorem 7.1):

An undirected bipartite graph $G = (X\dot{\cup}Y, E)$ has a total matching iff for all subsets $X_1 \subseteq X$ the property $|X_1| \leq |E(X_1)|$ holds.

Proof: The necessity of $|\rho| \leq |\rho\alpha$ is easy to see: let f be a total matching of α. Then f is an injection with the property $f \sqsubseteq \alpha$ and it holds:

$$
\begin{aligned}
|\rho| &= |\rho f f^\sharp| && \{f f^\sharp = id_X\} \\
&= |\rho f| && \{\text{ corollary 2.2(d) }\} \\
&\leq |\rho\alpha| && \{f \sqsubseteq \alpha\}
\end{aligned}
$$

So it is shown, that the requirement $|\rho| \leq |\rho\alpha|$ is necessary for the existence of a total matching $f \sqsubseteq \alpha$.

To show that the requirement $|\rho| \leq |\rho\alpha|$ is also sufficient we have to make much more effort. The rough strategy of our proof will be to embed the to sets X and Y in a network by joining X and Y and adding two nodes s and t. As edges in the new network we choose the edges between the nodes in X and Y given by α and introduce edges (s, x) for all nodes x in X and (y, t) for all nodes y in Y. All such edges are given the capacity one, so that there exists a two-valued maximal flow on it. From such a flow we will construct a total matching of α.

So let $X \cap Y = \emptyset$, let $\alpha : X \rightharpoondown Y$ be a boolean relation with $|\rho_0| \leq |\rho_0\alpha|$ for all boolena relations $\rho_0 : I \rightharpoondown X$ and let w.l.o.g. $(X \cup Y) \cap \{s, t\} = \emptyset$. We construct now a set $\hat(X)$ by $\hat{X} = \{s\}\dot\cup X\dot\cup Y\dot\cup\{t\}$ (the nodes of the network sketched above); we introduce bijective mappings $s : I \to \hat{X}$, $i : X \to \hat{X}$, $j : Y \to \hat{X}$ and $t : I \to \hat{X}$ with $s(*) = s$, $t(*) = t$, $i(x) = x$ and $j(y) = y$. Therefore hold $ss^\sharp = tt^\sharp = id_I$, $ii^\sharp = id_X$, $jj^\sharp = id_Y$ and $s^\sharp s \sqcup t^\sharp t \sqcup i^\sharp i \sqcup j^\sharp j = id_{\hat{X}}$. We construct a new boolean relation $\hat{\alpha} : \hat{X} \rightharpoondown \hat{X}$ by $\hat{\alpha} = s^\sharp \nabla_{IX} i \sqcup i^\sharp \alpha j \sqcup j^\sharp \nabla_{YI} t$ (the edges of the network). Next we show $\hat{\alpha} \sqcap \hat{\alpha}^\sharp = 0_{\hat{X}\hat{X}}$ by calculating

$$
\begin{aligned}
\hat{\alpha} \sqcap \hat{\alpha}^\sharp &= (s^\sharp \nabla_{IX} i \sqcup i^\sharp \alpha j \sqcup j^\sharp \nabla_{YI} t) \sqcap (s^\sharp \nabla_{IX} i \sqcup i^\sharp \alpha j \sqcup j^\sharp \nabla_{YI} t)^\sharp \\
&= (s^\sharp \nabla_{IX} i \sqcup i^\sharp \alpha j \sqcup j^\sharp \nabla_{YI} t) \sqcap (i^\sharp \nabla_{XI} s \sqcup j^\sharp \alpha^\sharp i \sqcup t^\sharp \nabla_{IY} j) \\
&= (s^\sharp \nabla_{IX} i \sqcap i^\sharp \nabla_{XI} s) \sqcup (s^\sharp \nabla_{IX} i \sqcap j^\sharp \alpha^\sharp i) \sqcup (s^\sharp \nabla_{IX} i \sqcap t^\sharp \nabla_{IY} j) \sqcup \\
&\quad (i^\sharp \alpha j \sqcap i^\sharp \nabla_{XI} s) \sqcup (i^\sharp \alpha j \sqcap j^\sharp \alpha^\sharp i) \sqcup (i^\sharp \alpha j \sqcap t^\sharp \nabla_{IY} j) \sqcup \\
&\quad (j^\sharp \nabla_{YI} t \sqcap i^\sharp \nabla_{XI} s) \sqcup (j^\sharp \nabla_{YI} t \sqcap j^\sharp \alpha^\sharp i) \sqcup (j^\sharp \nabla_{YI} t \sqcap t^\sharp \nabla_{IY} j)
\end{aligned}
$$

and convince ourselves, that every bracket yields zero: terms of the form $a^\sharp bc \sqcap d^\sharp ef$ with $a \neq d$ are zero because of the construction of $\hat(X)$, s, t, i and j, due to the fact, that every pair $\in a^\sharp bc$ on the first position has an element of a set disjoint to the set, from which the first entries of all pairs of $d^{sharp}ef$ are drawn. So there remain the two brackets $(i^\sharp \alpha j \sqcap i^\sharp \nabla_{XI} s)$ und $(j^\sharp \nabla_{YI} t \sqcap j^\sharp \alpha^\sharp i)$; here suffices the same argument, applied on the second entries.

From $\hat{\alpha} \sqcap \hat{\alpha}^\sharp = 0_{\hat{X}\hat{X}}$ we can conclude, that $N = (\hat{\alpha} : \hat{X} \rightharpoondown \hat{X}, s, t)$ is a network, which has according to theorem 6.6 a boolean maximal flow. Our goal will be to construct from this flow $\hat{\varphi} : \hat{X} \rightharpoondown \hat{X}$ a total matching $\varphi : X \rightharpoondown Y$.

The first step is to show, that $s : I \rightharpoondown \hat(X)$ is a minimal cut in (N). Therefore we observe, that every cut $\rho : I \rightharpoondown \hat(X)$ can be written as $\rho = s \sqcup \rho_0 i \sqcup \rho_1 j$ with suited boolean relations $\rho_0 : I \rightharpoondown X$ and $\rho_1 : I \rightharpoondown Y$. For its complement ρ^- holds $\rho^- = \rho_0^- i \sqcup \rho_1^- j \sqcup t$; so we can calculate:

$$
\begin{aligned}
|\hat{\alpha} \sqcap \rho^\sharp \rho^-| &= (s^\sharp \nabla_{IX} i \sqcup i^\sharp \alpha j \sqcup j^\sharp \nabla_{YI} t) \sqcap (s^\sharp \sqcup i^\sharp \rho_0^\sharp \sqcup j^\sharp \rho_1^\sharp)(\rho_0^- i \sqcup \rho_1^- j \sqcup t) \\
&= (s^\sharp \nabla_{IX} i \sqcup i^\sharp \alpha j \sqcup j^\sharp \nabla_{YI} t) \sqcap
\end{aligned}
$$

$$
\begin{aligned}
&(s^\sharp(\rho_0^- i \sqcup \rho_1^- j \sqcup t) \sqcup \\
&i^\sharp \rho_0^\sharp(\rho_0^- i \sqcup \rho_1^- j \sqcup t) \sqcup \\
&j^\sharp \rho_1^\sharp(\rho_0^- i \sqcup \rho_1^- j \sqcup t)) \\
= &(s^\sharp \nabla_{IX} i \sqcap s^\sharp(\rho_0^- i \sqcup \rho_1^- j \sqcup t)) \sqcup \\
&(i^\sharp \alpha j \sqcap i^\sharp \rho_0^\sharp(\rho_0^- i \sqcup \rho_1^- j \sqcup t)) \sqcup \\
&(j^\sharp \nabla_{YI} t \sqcap j^\sharp \rho_1^\sharp(\rho_0^- i \sqcup \rho_1^- j \sqcup t)) \\
= &s^\sharp \rho_0 i \sqcup i^\sharp(\alpha \sqcap \rho_0^\sharp \rho_0^-) j \sqcup j^\sharp \rho_1^\sharp t
\end{aligned}
$$

The arguments for omitting terms are the same as before; we won't repeat them here.

Now we can estimate the capacity of the cut ρ:

$$
\begin{aligned}
|\hat{\alpha} \sqcap \rho^\sharp \rho^-| &= |s^\sharp \rho_0 i \sqcup i^\sharp(\alpha \sqcap \rho_0^\sharp \rho_0^-) j \sqcup j^\sharp \rho_1^\sharp t| \\
&\qquad \{ \text{ disjointness } \} \\
&= |s^\sharp \rho_0 i| + |i^\sharp(\alpha \sqcap \rho_0^\sharp \rho_0^-) j| + |j^\sharp \rho_1^\sharp t| \\
&\qquad \{ \ s^\sharp, i, i^\sharp, j, j^\sharp, t \text{ univalent } \} \\
&= |\rho_0^-| + |\alpha \sqcap \rho_0^\sharp \rho_0^-| + |\rho_1^\sharp| \\
&\qquad \{ \text{ Dedekind inequality, } |\alpha^\sharp| = |\alpha| \ \} \\
&\geq |\rho_0^-| + |\rho_0 \alpha \sqcap \rho_1^-| + |\rho_1| \\
&\qquad \{ \text{ Disjointness } \} \\
&= |\rho_0^-| + |(\rho_0 \alpha \sqcap \rho_1^-) \sqcup \rho_1| \\
&\qquad \{ \ \rho_0 \alpha \sqsubseteq (\rho_0 \alpha \sqcap \rho_1^-) \sqcup \rho_1 \ \} \\
&\geq |\rho_0^-| + |\rho_0 \alpha| \\
&\qquad \{ \text{ requirement! } \} \\
&\geq |\rho_0^-| + |\rho_0| \\
&\qquad \{ \text{ Disjointness of } \rho^- \text{ and } \rho \ \} \\
&= |\rho_0^- \sqcup \rho_0| \\
&= |\nabla_{IX}|
\end{aligned}
$$

The capacity of the cut s is given by

$$
\begin{aligned}
|\hat{\alpha} \sqcap s^\sharp s^-| &= |s \hat{\alpha} \sqcap s^-| && \{ \ s \text{ univalent } \} \\
&= |\nabla_{IX} i \sqcap (\nabla_{IX} i \sqcup \nabla_{IY} j \sqcup t)| && \{ \text{ definition of } s, i, j, t \ \} \\
&= |\nabla_{IX} i| \\
&= |\nabla_{IX}| && \{ \ i \text{ univalent } \}
\end{aligned}
$$

So s is a minimal cut in N, because the capacity of every other cut is $\geq \nabla_{IX}$, as shown above.

For our desired matching we choose the relation $\varphi := i \hat{\varphi} j^\sharp : X \rightharpoondown Y$. Obviously holds $\varphi \sqsubseteq \alpha$. To show that φ is indeed a total matching we need two help claims, namely

(a) $\hat{\varphi}(s, xi) = 1$ for all point relations $x : I \rightharpoondown X$

and

(b) $i\hat{\varphi}jj^\sharp = i\hat{\varphi}$

To show (a) we calculate

$$
\begin{aligned}
|\nabla_{IX}| &= |\hat{\alpha} \sqcap s^\sharp s^-| && \{ \text{ see above } \}\\
&= val(\hat{\varphi}) && \{ \text{ } s \text{ minimal cut } \}\\
&= |s\hat{\varphi}| && \{ \text{ } val(\varphi) = |s\varphi \text{ } \}\\
&= |s\hat{\varphi} \sqcap \nabla_{IX} i| && \{ \text{ } s\hat{\varphi} \sqsubseteq s\hat{\alpha} \sqsubseteq \nabla_{IX} i \text{ } \}\\
&= |s\hat{\varphi} i^\sharp| && \{ \text{ } i \text{ matching } \}\\
&= \textstyle\sum_{x \in X} |s\hat{\varphi} i^\sharp x^\sharp|\\
&= \textstyle\sum_{x \in X} \hat{\varphi}(s, xi)
\end{aligned}
$$

Because of $|\nabla_{IX}| = |X|$ and because the values of $\hat{\varphi}$ are either zero or one every summand in the last sum has to be one, so that the claim follows.

In case (b) we observer, that according to the definitions of i and j and under consideration of the flow property $\hat{\varphi} \sqsubseteq \hat{\alpha}$ the following inclusions hold:

$$i\hat{\varphi} = i\hat{\varphi} \sqcap \alpha j \sqsubseteq (i\hat{\varphi} j^\sharp \sqcap \alpha) j \sqsubseteq i\hat{\varphi} j^\sharp j \sqsubseteq i\hat{\varphi}$$

Therefrom the claim follows.

To be a total matching φ has to be total, univalent and injective. The first two requirements are equivalent to

$$|x\varphi| = 1 \text{ for all point relations } x : I \rightharpoondown X,$$

whereas the last requirement can be described by the univalency of $\varphi^\sharp$. The desired properties of φ we will show in this form.

For the first we choose an arbitrary point relation $x : I \rightharpoondown X$ and calculate

$$
\begin{aligned}
|x\varphi| &= |xi\hat{\varphi} j^\sharp| && \{ \text{ definition of } \varphi \text{ } \}\\
&= |xi\hat{\varphi} j^\sharp j| && \{ \text{ } j \text{ injective } \}\\
&= |xi\hat{\varphi}| && \{ \text{ help claim (b) } \}\\
&= |\hat{\varphi}(xi)^\sharp| && \{ \text{ flow conservation } \}\\
&= |(s^\sharp s \sqcup i^\sharp i \sqcup j^\sharp j \sqcup t^\sharp t)\hat{\varphi}(xi)^\sharp| && \{ \text{ } id_{\hat{X}} = (s^\sharp s \sqcup i^\sharp i \sqcup j^\sharp j \sqcup t^\sharp t \text{ } \}\\
&= |s\hat{\varphi}(xi)^\sharp| && \{ \text{ see below } \}\\
&= \hat{\varphi}(s, xi) && \{ \text{ trivial } \}\\
&= 1 && \{ \text{ help claim (a) } \}
\end{aligned}
$$

We owe the reason for the step from the fifth to the sixth line: $\hat{\varphi}$ contains by construction only pairs in $\{s\} \times X \cup X \times Y \cup Y \times \{t\}$ whereas $(xi)^\sharp$ consists only of pairs $\in X \times Y$. That a relation $\alpha\hat{\varphi}(xi)^\sharp$ becomes nonempty, α has therefore to contain at least one pair, which contains at the second position an element of X.

The only possible relation in this case is $s^\sharp s$. As a intermediate result between line five and six we obtain the expression $|s^\sharp s\hat{\varphi}(xi)^\sharp|$, which we can simplify to the sixth line due to the matching properties of $s^\sharp$.

The univalency of $\varphi^\sharp$ is shown by proving the inequality $|\varphi y^\sharp| \leq 1$ for arbitrary point relations $y : I \rightharpoondown Y$:

$$
\begin{aligned}
|\varphi y^\sharp| &= |i\hat{\varphi}j^\sharp y^\sharp| && \{ \text{ definition of } \varphi \ \} \\
&= |i^\sharp i\hat{\varphi}(yj)^\sharp| && \{ \ i \text{ injective } \} \\
&\leq |\hat{\varphi}(yj)^\sharp| && \{ \ i^\sharp i \sqsubseteq id_{hatX} \ \} \\
&= |yj\hat{\varphi}| && \{ \text{ flow conservation } \} \\
&\leq |t| && \{ \ yj\hat{\varphi} \sqsubseteq yj\hat{\alpha} = y\nabla_{YI}t \sqsubseteq t \ \} \\
&= 1
\end{aligned}
$$

This completes the proof.

7.3 Generalisation of Hall's Theorem

Hall's theorem deals only with the situation, if $|\rho| \leq |\rho\alpha|$ for all point relations $\rho : I \rightharpoondown X$ holds. The following theorem makes it possible to predict what will happen in the case of "defect" relations, which don't satisfy this requirement properly.

Theorem 7.4: *Let $\alpha : X \rightharpoondown Y$, X and Y like in Hall's theorem and let $\delta(\alpha) := max_{\rho:I\rightharpoondown X}(|\rho| - |\rho\alpha|)$ like in Corollary 7.2. If $\delta(\alpha) > 0$, then exists a maximal matching f of α with $|f| = |\nabla_{XI}| - \delta(\alpha)$.*

Proof: We choose a set Z with $|Z| = \nabla_{IZ} = \delta(\alpha)$ und $Z \cap (X \cup Y) = \emptyset$, further similarly to the proof of Hall's theorem tow relations $i : Y \rightharpoondown Y\dot{\cup}Z$ and $j : Z \rightharpoondown Y\dot{\cup}Z$ with $i(y,x) = 1$ iff $y = x$ and $j(z,x,) = 1$ iff $z = x$. Additionally we construct a relation $\hat{\alpha} : X \rightharpoondown Y\dot{\cup}Z$ by $\hat{\alpha} = \alpha i \sqcup \nabla_{XZ}j$. Then holds for all relations $\rho : I \rightharpoondown X$ with $\rho \neq 0_{IX}$:

$$
\begin{aligned}
|\rho\hat{\alpha}| &= |\rho(\alpha i \sqcup \nabla_{XZ}j)| && \{ \text{ definition of } \hat{\alpha} \ \} \\
&= |\rho\alpha i| + |\rho\nabla_{XZ}j| && \{ \text{ disjointness } \} \\
&= |\rho\alpha| + |\rho\nabla_{XZ}j| && \{ \ |\rho\alpha i| = |\rho\alpha i i^\sharp| = |\rho\alpha| \ \} \\
&= |\rho\alpha| + |\rho\nabla_{XZ}| && \{ \ j \text{ matching } \} \\
&= |\rho\alpha| + |\nabla_{IZ}| && \{ \ \rho \neq 0_{IX} \Rightarrow \rho\nabla_{XZ} = \nabla_{IZ} \ \} \\
&= |\rho\alpha| + \delta(\alpha) && \{ \text{ construction of } Z \ \} \\
&\geq |\rho\alpha| + (|\rho| - |\rho\alpha|) && \{ \text{ definition of } \delta(\alpha) \ \} \\
&= |\rho|
\end{aligned}
$$

For $\rho = 0_{IX}$ obviously $|\rho| = |\rho\alpha| = 0$ holds, so Hall's theorem states here, that a injection $g : X \rightharpoondown Y$ with $g \sqsubseteq \hat{\alpha}$ exists. We now construct a relation $f := gi^\sharp : X \rightharpoondown Y$. This relation f is a matching, because

$$f^\sharp f = (gi^\sharp)^\sharp(gi^\sharp) = ig^\sharp gi^\sharp \sqsubseteq ii^\sharp = id_Y$$

and

$$ff^\sharp = (gi^\sharp)(gi^\sharp)^\sharp = gi^\sharp i g^\sharp \sqsubseteq gg^\sharp = id_X$$

hold.

From the injectivity of g and by construction of i and j follows

$$|\nabla_{IX}| = |g| = |g(i^\sharp i \sqcup j^\sharp j)| = |gi^\sharp| + |gj^\sharp|$$

and hence

$$|f| = |gi^\sharp| = |\nabla_{IX}| - |gj^\sharp| \geq |\nabla_{IX}| - |\nabla_{IZ}| = |\nabla_{IX}| - \delta(\alpha)$$

By corollary 7.2 holds $|f| \leq |\nabla_{IX}| - \delta(\alpha)$, so that both the desired equality and the maximality of f follow. ■